# A CONSTRUCTION OF RINGS WHOSE INJECTIVE HULLS ALLOW A RING STRUCTURE

**Dedicated to the memory of Hanna Neumann**

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In her paper [3], Osofsky exhibited an example of a ring $R$ containing 16 elements which (i) is equal to its left complete ring of quotients, (ii) is not self-injective and (iii) whose injective hull $HR = H({}_RR)$ allows a ring structure extending the $R$-module structure of $HR$. In the present note, we offer a general method of constructing such rings; in particular, given a non-trivial split Frobenius algebra $A$ and a natural $n \geqq 2$, a certain ring of $n \times n$ matrices over $A$ provides such an example. Here, taking for $A$ the semi-direct extension of $\mathbb{Z}/2\mathbb{Z}$ by itself and $n = 2$, one gets the example of Osofsky. Thus, our approach answers her question on finding a non-computational method for proving the existence of such rings.

Throughout the present note, $A$ denotes a ring with unity 1. Given an $A$-module $M$, denote by $\operatorname{Rad} M$ the intersection of all maximal submodules of $M$. Dually, if $M$ has minimal submodules, $\operatorname{Soc} M$ denotes their union. Also, write $\operatorname{Top} M = M/\operatorname{Rad} M$. The radical $\operatorname{Rad} A$ of the ring $A$ will be denoted consistently by $W$ and the factor $A/W$ by $Q$. By a *split ring* $A$ we shall understand a ring which is a semi-direct extension $(Q, W)$ of $W$ by $Q$; in this case, we shall consider $Q$ to be embedded as a subring in $A$. Thus $A = Q \oplus W$ as additive groups and $(q_1, w_1)(q_2, w_2) = (q_1q_2, q_1w_2 + w_1q_2 + w_1w_2)$. For example, it is well-known that every finite dimensional algebra over an algebraically closed field is a split ring.

We recall that a *Frobenius algebra* $A$ is a finite dimensional algebra over a field $F$ which is self-injective; and that, given a decomposition $A = \oplus_{i=1}^{s} Ae_i$ into indecomposable left ideals, there exists a permutation $\pi$ of $\{1, 2, \cdots, s\}$ such that $\operatorname{Soc} Ae_i \cong \operatorname{Top} Ae_{\pi(i)}$.

Given a ring $R$ and an $R$-module $M$, the injective hull of $M$ will be denoted by $HM$, the injective hull of ${}_RR$ by $HR$. The double centralizer of $HR$ is called the

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*left complete ring of quotients* of $R$ (cf. [2]). An essential extension $M$ of the ring $R$, i.e. a left $R$-module $M$ containing ${}_RR$ as an essential submodule, is said *to allow a ring structure*, if $M$ can be made into a ring in such a way that the ring multiplication extends the given $R$-module multiplication.

Let $A = U \oplus V$ be a semi-direct extension of the two-sided ideal $V$ by the subring $U$ of $A$. In what follows, we shall consider, for a given $n \geqq 2$, a subring $R$ of the ring $A_n$ of all $n \times n$ matrices over $A$. The subring $R = R(U \oplus V, n) = UI + T$, where $I$ denotes the $n \times n$ identity matrix and

$$T = \{(a_{ij}) \in A_n \mid a_{ij} = 0 \text{ for } i \geqq 2,\ a_{11} \in V\}.$$

LEMMA 1. *Let $A = U \oplus V$ be a semi-direct extension of $V$ by $U$ such that, for every non-zero $u \in U$, $Vu \neq 0$. Then $A_n$ (considered as a left $R$-module) is an essential extension of the ring $R = R(U \oplus V, n)$.*

PROOF. Throughout the proof, the matrix $J_{kl} = (x_{ij}) \in A_n$ is defined by $x_{kl} = 1$ and $x_{ij} = 0$ otherwise.

Take $0 \neq (a_{ij}) \in A_n$. If $a_{ij} \neq 0$ for $i \geqq 2$, then $J_{1i} \in R$ and

$$(b_{ij}) = J_{1i}(a_{ij}) \in A_n$$

is a non-zero matrix with $b_{ij} = 0$ for all $i \geqq 2$. Let $b_{11} = u + v$ with $u \in U$ and $v \in V$. If $u = 0$, then $(b_{ij}) \in R$ and the proof is done. If $u \neq 0$, then there is $v' \in V$ such that $v'u \neq 0$, and thus $v'J_{11} \in R$ and

$$0 \neq (v'J_{11})(b_{ij}) = (v'J_{1i})(a_{ij}) \in R.$$

Lemma 1 follows.

REMARK. Observe that the preceding simple lemma provides a wide variety of rings with essential extensions which allow a ring structure.

LEMMA 2. *Let $A$ be a split ring which is left artinian and whose left socle contains simple left modules of all possible types. Then $R = R(Q \oplus W, n)$ is its left complete ring of quotients.*

PROOF. Let $M \subseteq R$ consist of all matrices $(a_{ij}) \in A_n$ with $a_{11} = 0$ and $a_{ij} = 0$ for $i \geqq 2$. Obviously, $M$ is a two-sided ideal of $R$ and can be considered as a left $A$-module ${}_AM$; in this way, the left ideals of $R$ contained in $M$ are just the submodules of ${}_AM$. Therefore every composition series of ${}_AM$ is also a composition series of ${}_RM$, and since $R/M$ and $A$ are isomorphic rings, $R$ is left artinian.

Furthermore, if $\{f_1, f_2, \cdots, f_s\}$ is an orthogonal set of primitive idempotents in $A$ whose sum is 1 and if

$$f_i = e_i + w_i \text{ with } e_i \in Q,\ w_i \in W \text{ for } i = 1, 2, \cdots, s,$$

then $\{e_1, e_2, \cdots, e_s\}$ is an orthogonal set of primitive idempotents whose sum is 1 contained in $Q$. Thus

$$\{E_1, E_2, \cdots, E_s\}, \text{ where } E_i = e_i I, \qquad i = 1, 2, \cdots, s,$$

is an orthogonal set of primitive idempotents in $R$ whose sum is $1 \in R$.

Now, put

$$P = \{(a_{ij}) \in T \mid a_{ij} \in \mathrm{Soc}_A A\};$$

one can see immediately that $P \subseteq \mathrm{Soc}_R R$. Since

$$e_i \,\mathrm{Soc}_A A \neq 0 \text{ if and only if } E_i P \neq 0,$$

we conclude, in view of our hypothesis on the left socle of $A$, that the left socle of $R$ contains simple left modules of all types. As a consequence, ${}_R R$ has no proper rational extension and since the left complete ring of quotients of $R$ is the maximal rational extension of ${}_R R$, Lemma 2 follows.

REMARK. Observe that the method of the proof of Lemma 2 enables to prove the assertion under the weaker assumption that the ring $A$ is right perfect.

The main result of our note reads as follows.

THEOREM. *Let $A$ be a two-sided indecomposable split Frobenius algebra with non-zero radical. Then $R = R(Q \oplus W, n)$ coincides with its left complete ring of quotients and $A_n$ is its left injective hull. Thus, the injective hull of $R$ allows a ring structure.*

PROOF. Let $A$ be finite dimensional over the field $F$. Since $A$ is a split Frobenius algebra, Lemma 2 yields immediately that $R$ coincides with its left complete ring of quotients. Furthermore, in a Frobenius algebra the left and right socles are equal and thus every element $u \in R$ such that $uW = 0$ belongs necessarily to Soc $A$. Also, if $\{e_1, e_2, \cdots, e_s\} \subseteq Q$ is an orthogonal set of primitive idempotents whose sum is $1 \in A$, $We_i \neq 0$ for all $i$; for, otherwise, the direct sum of all $Ae_i$ such that $We_i = 0$ is a proper two-sided direct summand of $A$. Consequently,

$$\mathrm{Soc}\, A = \bigoplus_{i=1}^{s} \mathrm{Soc}\, Ae_i \subseteq \bigoplus_{i=1}^{s} \mathrm{Rad}\, Ae_i = W.$$

In view of this inclusion, we can apply Lemma 1 and obtain that $A_n$ is an essential extension of ${}_R R$.

Now, writing $E_i = e_i I$, we have

$$(qI + t)E_i = qe_i I + te_i \text{ for every } q \in Q \text{ and } t \in T.$$

Thus, if $\pi$ is a permutation of $\{1, 2, \cdots, s\}$ such that

$$\mathrm{Soc}\, Ae_i \cong \mathrm{Top}\, Ae_{\pi(i)},$$

we deduce that Soc $RE_i$ is a direct sum of $n$ copies of Top $RE_{\pi(i)}$. For,

$$\operatorname{Soc} RE_i = \{(a_{ij}) \in A_n \mid a_{ij} \in \operatorname{Soc} Ae_i \text{ and } a_{ij} = 0 \text{ for } i \geqq 2\}$$

is of length $n$ and, obviously, no simple submodule of $RE_i$ is annihilated by $E_{\pi(i)}$. Hence

$$HR = \bigoplus_{i=1}^{s} I(\operatorname{Soc} RE_i) = \bigoplus_{i=1}^{s} \bigoplus_{j=1}^{n} H(\operatorname{Top} RE_{\pi(i)}).$$

Now, since

$$H(\operatorname{Top} RE_{\pi(i)}) \cong \operatorname{Hom}_F(E_{\pi(i)}R, F)$$

(cf. [1]), we calculate

$$\dim_F HR = \sum_{i=1}^{s} \sum_{j=1}^{n} \dim_F H(\operatorname{Top} RE_{\pi(i)}) = n \sum_{i=1}^{s} \dim_F (E_{\pi(i)}R) = n \dim_F R,$$

because $\pi$ is a permutation and thus $\oplus_{i=1}^{s} E_{\pi(i)} R = R$. Furthermore, by the definition of $R$

$$\dim_F R = n \dim_F A,$$

and consequently,

$$\dim_F \mathrm{HR} = n \dim_F R = n^2 \dim_F A = \dim_F (A_n),$$

as required.

The proof of Theorem is completed.

EXAMPLE. For every field $F$, the split extension $A = (F, F)$ of $F$ by itself (with the multiplication $(f_1, f_2)(f_1', f_2') = (f_1 f_1', f_1 f_2' + f_2 f_1')$) is a Frobenius algebra which satisfies the assumptions of the Theorem. Thus, in this way, we get rings whose injective hulls allow a ring structure. If we take $F = \mathbb{Z}/2\mathbb{Z}$ and $n = 2$, we obtain the example of Osofsky [3]. Here, the radical of $A = (\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ is given by $W = \{0, \omega\}$, whereas $Q = \{0, \varepsilon\}$ with $0 = (0,0)$, $\omega = (0,1)$ and $\varepsilon = (1,0)$. Since only right modules are considered in [3], the corresponding ring is given by

$$R = \left\{ \begin{pmatrix} q + w & 0 \\ a & q \end{pmatrix} \,\middle|\, q \in Q, w \in W, a \in A \right\}.$$

It can be checked easily that the elements

$$1 = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}, \quad x = \begin{pmatrix} 0 & 0 \\ \varepsilon & 0 \end{pmatrix}, \quad y = \begin{pmatrix} \omega & 0 \\ 0 & 0 \end{pmatrix}, \quad xy = \begin{pmatrix} 0 & 0 \\ \omega & 0 \end{pmatrix}$$

generate $R$ additively, and that they satisfy the equalities

$$0 = x^2 = y^2 = (xy)^2 = yx = x(xy) = y(xy) = (xy)x = (xy)y.$$

Also, the remaining generators of the right injective hull of $R$ given in [3] can be identified with the following elements of $A_2$

$$m = \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix}, \quad n = \begin{pmatrix} \varepsilon & 0 \\ 0 & 0 \end{pmatrix}, \quad u = \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}, \quad \text{and } \bar{m} = \begin{pmatrix} 0 & 0 \\ 0 & \omega \end{pmatrix}.$$

REMARK. The local ring $R = (F \oplus F, 2)$ of the proceeding example can be easily shown to have the property that both its left and right injective hulls are isomorphic to $F_2$ (and that both the left and right injective hulls allow a ring structure). In fact, more generally, *if $A$ is a commutative two-sided indecomposable split Frobenius algebra with non-zero radical $W$ such that $W^2 = 0$, then the right injective hull of $R(Q \oplus W, 2)$ is isomorphic to $A_2$.* This follows immediately from the fact that, as a consequence of our assumptions, $A$ is a local ring and there is an anti-automorphism $\Phi$ of $R$ defined by

$$\begin{pmatrix} q+w & q'+w' \\ 0 & q \end{pmatrix} \Phi = \begin{pmatrix} q+q'\phi & w\phi^{-1}+w' \\ 0 & q \end{pmatrix}$$

with $q, q' \in Q$, $w, w' \in W$ and an isomorphism

$$\phi \colon {}_QQ_Q \to {}_QW_Q.$$

The assumptions of the above assertions are rather natural due to the following result: *If, under the assumptions of Theorem, the right injective hull $H(R_R)$ of $R = R(Q \oplus W, n)$ is isomorphic to $A_n$, then $n = 2$ and $W^2 = 0$.* For,

$$\operatorname{Soc} R_R = \{(a_{ij}) \in A_n \mid a_{11} = 0 \text{ and } a_{ij} = 0 \text{ for } i \geqq 2\}$$

and, following the notation of the proof of Theorem, one can see easily that $\operatorname{Soc} R_R$ is the direct sum of $(n-1)\cdot\partial(e_iA)$ copies of $\operatorname{Top} E_iR$ $(1 \leqq i \leqq s)$; here, $\partial(e_iA)$ denotes the (right) length of $e_iA$. Therefore,

$$\begin{aligned} \dim_F H(R_R) &= \sum_{i=1}^{s} (n-1)\partial(e_iA)\cdot\dim_F(RE_{\pi(i)}) \\ &= \sum_{i=1}^{s} (n-1)\partial(e_iA)\cdot n\dim_F(Ae_{\pi(i)}), \end{aligned}$$

and thus, since $\dim_F(A_n) = n^2 \dim_F A$,

$$n(n-1)\sum_{i=1}^{s} \partial(e_iA)\dim_F(Ae_{\pi(i)}) = n^2 \dim_F A.$$

Using the fact that $\partial(e_iA) \geqq 2$ for all $1 \leqq i \leqq s$, one gets that

$$2(n-1)\dim_F A \leqq n\dim_F A,$$

and thus $n \leqq 2$. Consequently, $n = 2$ and hence $\partial(e_i A) = 2$ for all $1 \leqq i \leqq s$, i.e. $W^2 = 0$, as required.

We recall that the subring $B$ of the ring $A$ is called a *left order* in $A$, if every element of $A$ can be written in the form $b^{-1}b'$ with elements $b$ and $b'$ from $B$.

COROLLARY. *Let $A$ be a two-sided indecompsable split Frobenius algebra with non-zero radical. Let $B$ be a left order of $A$ such that $B = U \oplus V$ (as additive groups) with $U \subseteq Q$ and $V \subseteq W$. Then* the ring $R = R(Q \oplus W, n)$ is the left complete ring of quotients of $S = R(U \oplus V, n)$ *and $A_n$ is the left injective hull of $S$. Thus, the injective hull of $S$ allows a ring structure.*

PROOF. First, we show that $S$ is a left order in $R$. Given $(a_{ij}) \in R$, we can find elements $b_j$ and $b \in B$ such that

$$b^{-1}b_j = a_{1j} \text{ for } 1 \leqq j \leqq n.$$

Let $b = u + v$ and $b_1 = u_1 + v_1$ with $u,\ u_1 \in U$ and $v,\ v_1 \in V$. Furthermore, we can write

$$b^{-1} = q + w \text{ with } q \in Q \text{ and } w \in W.$$

One can see easily that $q$ is the inverse of $u$ in $A$. Consequently, the matrix $qI + wJ_{11} \in R$ is the inverse of the matrix $uI + vJ_{11} \in S$. Also, the equality $b^{-1}b_1 = a_{11}$ together with the fact that $a_{11} = a_{jj} + w'$ $(2 \leqq j \leqq n)$, for some $w' \in W$, implies

$$qu_1 + (qv_1 + wu_1 + wv_1) = (q + w)(u_1 + v_1) = a_{jj} + w',$$

and thus $qu_1 = a_{jj}$. Therefore, setting

$$B = u_1 I + t \text{ with } t = (b_{ij}) \in T, \text{ where } b_{11} = b - u_1, \text{ and } b_{1j} = b_j \text{ for } 2 \leqq j \leqq n,$$

$$(uI + vJ_{11})^{-1}B = (qI + wJ_{11})B = (a_{ij}),$$

and since both matrices $uI + vJ_{11}$ and $B$ belong to $S$, $S$ is a left order in $R$, as required.

Now, it is well-known that, for a left order $S$ in $R$, ${}_SR$ is a rational extension of ${}_SS$ and that every rational extension of ${}_SS$ containing $R$ is also a rational extension of ${}_RR$. Therefore, according to Theorem, ${}_SR$ is the maximal rational extension of ${}_SS$, i.e $R$ is the left complete ring of quotients of $S$. And, since $A_n$ is obviously the injective hull of ${}_SS$, our proof is completed.

EXAMPLE. The split extension $B = (\mathbb{Z}, \mathbb{Z})$ of $\mathbb{Z}$ by itself is a left order in $A = (\mathbb{Q}, \mathbb{Q})$ which satisfies the assumptions of Corollary.

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## References

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