A CONSTRUCTION OF RINGS WHOSE INJECTIVE HULLS ALLOW A RING STRUCTURE

Dedicated to the memory of Hanna Neumann

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In her paper [3], Osofsky exhibited an example of a ring $R$ containing 16 elements which (i) is equal to its left complete ring of quotients, (ii) is not self-injective and (iii) whose injective hull $HR = H(RR)$ allows a ring structure extending the $R$-module structure of $HR$. In the present note, we offer a general method of constructing such rings; in particular, given a non-trivial split Frobenius algebra $A$ and a natural $n \geq 2$, a certain ring of $n \times n$ matrices over $A$ provides such an example. Here, taking for $A$ the semi-direct extension of $Z/2Z$ by itself and $n = 2$, one gets the example of Osofsky. Thus, our approach answers her question on finding a non-computational method for proving the existence of such rings.

Throughout the present note, $A$ denotes a ring with unity 1. Given an $A$-module $M$, denote by $\text{Rad} M$ the intersection of all maximal submodules of $M$. Dually, if $M$ has minimal submodules, $\text{Soc} M$ denotes their union. Also, write $\text{Top} M = M / \text{Rad} M$. The radical $\text{Rad} A$ of the ring $A$ will be denoted consistently by $W$ and the factor $A/W$ by $Q$. By a split ring $A$ we shall understand a ring which is a semi-direct extension $(Q, W)$ of $W$ by $Q$; in this case, we shall consider $Q$ to be embedded as a subring in $A$. Thus $A = Q \oplus W$ as additive groups and $(q_1, w_1) (q_2, w_2) = (q_1 q_2, q_1 w_2 + w_1 q_2 + w_1 w_2)$. For example, it is well-known that every finite dimensional algebra over an algebraically closed field is a split ring.

We recall that a Frobenius algebra $A$ is a finite dimensional algebra over a field $F$ which is self-injective; and that, given a decomposition $A = \oplus_{i=1}^s Ae_i$ into indecomposable left ideals, there exists a permutation $\pi$ of $\{1, 2, \cdots, s\}$ such that $\text{Soc} Ae_i \cong \text{Top} Ae_{\pi(i)}$.

Given a ring $R$ and an $R$-module $M$, the injective hull of $M$ will be denoted by $HM$, the injective hull of $RR$ by $HR$. The double centralizer of $HR$ is called the

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left complete ring of quotients of $R$ (cf. [2]). An essential extension $M$ of the ring $R$, i.e. a left $R$-module $M$ containing $rR$ as an essential submodule, is said to allow a ring structure, if $M$ can be made into a ring in such a way that the ring multiplication extends the given $R$-module multiplication.

Let $A = U \oplus V$ be a semi-direct extension of the two-sided ideal $V$ by the subring $U$ of $A$. In what follows, we shall consider, for a given $n \geq 2$, a subring $R$ of the ring $A_n$ of all $n \times n$ matrices over $A$. The subring $R = R(U \oplus V, n) = UI + T$, where $I$ denotes the $n \times n$ identity matrix and

$$T = \{(a_{ij}) \in A_n | a_{ij} = 0 \text{ for } i \geq 2, \ a_{11} \in V\}.$$

Lemma 1. Let $A = U \oplus V$ be a semi-direct extension of $V$ by $U$ such that, for every non-zero $u \in U$, $Vu \neq 0$. Then $A_n$ (considered as a left $R$-module) is an essential extension of the ring $R = R(U \oplus V, n)$.

Proof. Throughout the proof, the matrix $J_{kl} = (x_{ij}) \in A_n$ is defined by $x_{kl} = 1$ and $x_{ij} = 0$ otherwise.

Take $0 \neq (a_{ij}) \in A_n$. If $a_{ij} \neq 0$ for $i \geq 2$, then $J_{11} \in R$ and

$$(b_{ij}) = J_{11}(a_{ij}) \in A_n$$

is a non-zero matrix with $b_{ij} = 0$ for all $i \geq 2$. Let $b_{11} = u + v$ with $u \in U$ and $v \in V$. If $u = 0$, then $(b_{ij}) \in R$ and the proof is done. If $u \neq 0$, then there is $v' \in V$ such that $v'u \neq 0$, and thus $v' J_{11} \in R$ and

$$0 \neq (v' J_{11}) (b_{ij}) = (v' J_{11}) (a_{ij}) \in R.$$  

Lemma 1 follows.

Remark. Observe that the preceding simple lemma provides a wide variety of rings with essential extensions which allow a ring structure.

Lemma 2. Let $A$ be a split ring which is left artinian and whose left socle contains simple left modules of all possible types. Then $R = R(Q \oplus W, n)$ is its left complete ring of quotients.

Proof. Let $M \subseteq R$ consist of all matrices $(a_{ij}) \in A_n$ with $a_{11} = 0$ and $a_{ij} = 0$ for $i \geq 2$. Obviously, $M$ is a two-sided ideal of $R$ and can be considered as a left $A$-module $\mathbb{A} M$; in this way, the left ideals of $R$ contained in $M$ are just the submodules of $\mathbb{A} M$. Therefore every composition series of $\mathbb{A} M$ is also a composition series of $R/M$, and since $R/M$ and $A$ are isomorphic rings, $R$ is left artinian.

Furthermore, if $\{f_1, f_2, \ldots, f_s\}$ is an orthogonal set of primitive idempotents in $A$ whose sum is 1 and if

$$f_i = e_i + w_i \text{ with } e_i \in Q, \ w_i \in W \text{ for } i = 1, 2, \ldots, s,$$
then \(\{e_1, e_2, \ldots, e_s\}\) is an orthogonal set of primitive idempotents whose sum is 1 contained in \(Q\). Thus
\[ \{E_1, E_2, \ldots, E_s\}, \text{ where } E_i = e_iI, \quad i = 1, 2, \ldots, s, \]
is an orthogonal set of primitive idempotents in \(R\) whose sum is \(1 \in R\).

Now, put
\[ P = \{(a_{ij}) \in T | a_{ij} \in \text{Soc}_A A\}; \]
one can see immediately that \(P \subseteq \text{Soc}_R R\). Since
\[ e_i \text{Soc}_A A \neq 0 \text{ if and only if } E_i P \neq 0, \]
we conclude, in view of our hypothesis on the left socle of \(A\), that the left socle of \(R\) contains simple left modules of all types. As a consequence, \(R\) has no proper rational extension and since the left complete ring of quotients of \(R\) is the maximal rational extension of \(R\), Lemma 2 follows.

**Remark.** Observe that the method of the proof of Lemma 2 enables to prove the assertion under the weaker assumption that the ring \(A\) is right perfect.

The main result of our note reads as follows.

**Theorem.** Let \(A\) be a two-sided indecomposable split Frobenius algebra with non-zero radical. Then \(R = R(Q \oplus W, n)\) coincides with its left complete ring of quotients and \(A_n\) is its left injective hull. Thus, the injective hull of \(R\) allows a ring structure.

**Proof.** Let \(A\) be finite dimensional over the field \(F\). Since \(A\) is a split Frobenius algebra, Lemma 2 yields immediately that \(R\) coincides with its left complete ring of quotients. Furthermore, in a Frobenius algebra the left and right socles are equal and thus every element \(u \in R\) such that \(uW = 0\) belongs necessarily to \(\text{Soc} A\). Also, if \(\{e_1, e_2, \ldots, e_s\} \subseteq Q\) is an orthogonal set of primitive idempotents whose sum is \(1 \in A\), \(We_i \neq 0\) for all \(i\); for, otherwise, the direct sum of all \(Ae_i\) such that \(We_i = 0\) is a proper two-sided direct summand of \(A\). Consequently,
\[ \text{Soc} A \supseteq \bigoplus_{i=1}^{s} \text{Soc} Ae_i \supseteq \bigoplus_{i=1}^{s} \text{Rad} Ae_i = W. \]

In view of this inclusion, we can apply Lemma 1 and obtain that \(A_n\) is an essential extension of \(R\).

Now, writing \(E_i = e_iI\), we have
\[ (qI + t)E_i = qe_iI + te_i \text{ for every } q \in Q \text{ and } t \in T. \]
Thus, if \(\pi\) is a permutation of \(\{1, 2, \ldots, s\}\) such that
\[ \text{Soc} Ae_i \cong \text{Top} Ae_{\pi(i)}, \]
we deduce that \( \text{Soc } RE_i \) is a direct sum of \( n \) copies of \( \text{Top } RE_{n(t)} \). For,

\[
\text{Soc } RE_i = \{(a_{ij}) \in A_n \mid a_{ij} \in \text{Soc } Ae_i \text{ and } a_{ij} = 0 \text{ for } i \geq 2\}
\]

is of length \( n \) and, obviously, no simple submodule of \( RE_i \) is annihilated by \( E_{n(t)} \).

Hence

\[
HR = \bigoplus_{i=1}^{s} I(\text{Soc } RE_i) = \bigoplus_{i=1}^{s} \bigoplus_{j=1}^{n} H(\text{Top } RE_{n(t)}).
\]

Now, since

\[
H(\text{Top } RE_{n(t)}) \cong \text{Hom}_F(E_{n(t)}R, F)
\]

(cf. [1]), we calculate

\[
\dim F HR = \sum_{i=1}^{s} \sum_{j=1}^{n} \dim F H(\text{Top } RE_{n(t)}) = n \sum_{i=1}^{s} \dim F(E_{n(t)}R) = n \dim F R,
\]

because \( \pi \) is a permutation and thus \( \bigoplus_{i=1}^{s} E_{n(t)} R = R \). Furthermore, by the definition of \( R \)

\[
\dim F R = n \dim F A,
\]

and consequently,

\[
\dim F HR = n \dim F R = n^2 \dim F A = \dim F (A_n),
\]

as required.

The proof of Theorem is completed.

**Example.** For every field \( F \), the split extension \( A = (F, F) \) of \( F \) by itself (with the multiplication \((f_1, f_2)(f'_1, f'_2) = (f_1 f'_1, f_1 f'_2 + f_2 f'_1)\)) is a Frobenius algebra which satisfies the assumptions of the Theorem. Thus, in this way, we get rings whose injective hulls allow a ring structure. If we take \( F = \mathbb{Z}/2\mathbb{Z} \) and \( n = 2 \), we obtain the example of Ososky [3]. Here, the radical of \( A = (\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \) is given by \( W = \{0, \omega\} \), whereas \( Q = \{0, e\} \) with \( 0 = (0, 0), \omega = (0, 1) \) and \( e = (1, 0) \). Since only right modules are considered in [3], the corresponding ring is given by

\[
R = \left\{ \begin{pmatrix} q + w & 0 \\ a & q \end{pmatrix} \mid q \in Q, w \in W, a \in A \right\}.
\]

It can be checked easily that the elements

\[
l = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}, \quad x = \begin{pmatrix} 0 & 0 \\ e & 0 \end{pmatrix}, \quad y = \begin{pmatrix} \omega & 0 \\ 0 & 0 \end{pmatrix}, \quad xy = \begin{pmatrix} 0 & 0 \\ \omega & 0 \end{pmatrix}
\]

generate \( R \) additively, and that they satisfy the equalities

\[
0 = x^2 = y^2 = (xy)^2 = yx = x(xy) = y(xy) = (xy)x = (xy)y.
\]
Also, the remaining generators of the right injective hull of \( R \) given in [3] can be identified with the following elements of \( A_2 \)

\[
m = \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix}, \quad n = \begin{pmatrix} \varepsilon & 0 \\ 0 & 0 \end{pmatrix}, \quad u = \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \tilde{m} = \begin{pmatrix} 0 & 0 \\ 0 & \omega \end{pmatrix}.
\]

**Remark.** The local ring \( R = (F \oplus F, 2) \) of the proceeding example can be easily shown to have the property that both its left and right injective hulls are isomorphic to \( F_2 \) (and that both the left and right injective hulls allow a ring structure). In fact, more generally, if \( A \) is a commutative two-sided indecomposable split Frobenius algebra with non-zero radical \( W \) such that \( W^2 = 0 \), then the right injective hull of \( R(Q \oplus W, 2) \) is isomorphic to \( A_2 \). This follows immediately from the fact that, as a consequence of our assumptions, \( A \) is a local ring and there is an anti-automorphism \( \Phi \) of \( R \) defined by

\[
\begin{pmatrix} q + w & q' + w' \\ 0 & q \end{pmatrix} \Phi = \begin{pmatrix} q & q' \phi \\ w \phi^{-1} + w' & q \end{pmatrix}
\]

with \( q, q' \in Q, w, w' \in W \) and an isomorphism

\[
\phi: qQq \rightarrow qWq.
\]

The assumptions of the above assertions are rather natural due to the following result: If, under the assumptions of Theorem, the right injective hull \( H(R_R) \) of \( R = R(Q \oplus W, n) \) is isomorphic to \( A_n \), then \( n = 2 \) and \( W^2 = 0 \). For,

\[
\text{Soc } R_R = \{(a_{ij}) \in A_n \mid a_{11} = 0 \text{ and } a_{ij} = 0 \text{ for } i \geq 2\}
\]

and, following the notation of the proof of Theorem, one can see easily that \( \text{Soc } R_R \) is the direct sum of \( (n - 1) \cdot \partial(e_iA) \) copies of \( \text{Top } E_iR \) \((1 \leq i \leq s)\); here, \( \partial(e_iA) \) denotes the (right) length of \( e_iA \). Therefore,

\[
\dim_F H(R_R) = \sum_{i=1}^{s} (n - 1)\partial(e_iA) \cdot \dim_F(Re_{i(0)})
\]

\[
= \sum_{i=1}^{s} (n - 1)\partial(e_iA) \cdot n \dim_F(Re_{i(0)}),
\]

and thus, since \( \dim_F(A_n) = n^2 \dim_F A \),

\[
n(n - 1) \sum_{i=1}^{s} \partial(e_iA) \dim_F(Re_{i(0)}) = n^2 \dim_F A.
\]

Using the fact that \( \partial(e_iA) \geq 2 \) for all \( 1 \leq i \leq s \), one gets that

\[
2(n - 1) \dim_F A \leq n \dim_F A,
\]
and thus \( n \leq 2 \). Consequently, \( n = 2 \) and hence \( \delta(e_iA) = 2 \) for all \( 1 \leq i \leq s \), i.e. \( W^2 = 0 \), as required.

We recall that the subring \( B \) of the ring \( A \) is called a \textit{left order} in \( A \), if every element of \( A \) can be written in the form \( b^{-1}b' \) with elements \( b \) and \( b' \) from \( B \).

**COROLLARY.** Let \( A \) be a two-sided indecomposable split Frobenius algebra with non-zero radical. Let \( B \) be a left order of \( A \) such that \( B = U \oplus V \) (as additive groups) with \( U \subseteq Q \) and \( V \subseteq W \). Then the ring \( R = R(Q \oplus W, n) \) is the left complete ring of quotients of \( S = R(U \oplus V, n) \) and \( A_n \) is the left injective hull of \( S \). Thus, the injective hull of \( S \) allows a ring structure.

**PROOF.** First, we show that \( S \) is a left order in \( R \). Given \( (a_{ij}) \in R \), we can find elements \( b_j \) and \( b \in B \) such that
\[
b^{-1}b_j = a_{1j} \quad \text{for} \quad 1 \leq j \leq n.
\]
Let \( b = u + v \) and \( b_1 = u_1 + v_1 \) with \( u, u_1 \in U \) and \( v, v_1 \in V \). Furthermore, we can write
\[
b^{-1} = q + w \quad \text{with} \quad q \in Q \text{ and } w \in W.
\]
One can see easily that \( q \) is the inverse of \( u \) in \( A \). Consequently, the matrix \( qI + wJ_{11} \in R \) is the inverse of the matrix \( uI + vJ_{11} \in S \). Also, the equality \( b^{-1}b_1 = a_{11} \) together with the fact that \( a_{11} = a_{jj} + w' \) \( (2 \leq j \leq n) \), for some \( w' \in W \), implies
\[
(qu_1 + (qv_1 + wu_1 + vw_1) = (q + w)(u_1 + v_1) = a_{jj} + w',
\]
and thus \( qu_1 = a_{jj} \). Therefore, setting
\[
B = u_1I + t \quad \text{with} \quad t = (b_{1j}) \in T, \quad \text{where} \quad b_{11} = b - u_1, \quad \text{and} \quad b_{1j} = b_j \quad \text{for} \quad 2 \leq j \leq n,
\]
\[
(uI + vJ_{11})^{-1}B = (qI + wJ_{11})B = (a_{ij}),
\]
and since both matrices \( uI + vJ_{11} \) and \( B \) belong to \( S \), \( S \) is a left order in \( R \), as required.

Now, it is well-known that, for a left order \( S \) in \( R \), \( sR \) is a rational extension of \( sS \) and that every rational extension of \( sS \) containing \( R \) is also a rational extension of \( sR \). Therefore, according to Theorem, \( sR \) is the maximal rational extension of \( sS \), i.e \( R \) is the left complete ring of quotients of \( S \). And, since \( A_n \) is obviously the injective hull of \( sS \), our proof is completed.

**EXAMPLE.** The split extension \( B = (Z, Z) \) of \( Z \) by itself is a left order in \( A = (Q, Q) \) which satisfies the assumptions of Corollary.

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Injective hulls allow a ring structure

References


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