MONOPOLISTIC QUANTITY RATIONING*

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In this paper we address the question of whether a price-setting monopolist can improve his welfare by imposing quantity constraints on buyers. We show first that if all buyers are identical, quantity rations are not useful for the monopolist. We then show by means of an example that if buyers are diverse, quantity rations may be desirable. It is shown that if there are only two commodities, the only constraint that may be useful to the monopolist is a zero constraint on one of the two commodities. An example shows that this property does not hold for more than two commodities.

This paper is an attempt to understand when an agent with market power can use quantity rationing, in addition to prices, to improve his welfare. Our analysis connects two recent literatures: the theory of allocation with fixed prices (see, e.g., Benassy [1975], Drèze [1975]), and the theory of nonlinear pricing by a monopolist (e.g., Spence [1977]).

The fixed price literature studies the role of quantity rations in equilibrating competitive markets where prices are rigid. Although the theory has attracted much interest, it continues to suffer from the criticism that it provides no explanation for *why* prices should be rigid (or move less flexibly than quantities).

The traditional theory of monopoly models a monopolist as an agent who sets prices to maximize his utility (or profit), given the price-taking behavior of the other agents in the economy. More elaborate models allow the monopolist to set price *schedules*, rules that specify price as a function of quantity. Nonlinear price schedules allow a monopolist to sell his goods discriminately in two ways; they enable the monopolist to capture consumer surplus of any *given* buyer, and they permit him to exploit differences in demand *across* consumers.

One especially simple form of nonlinear pricing consists of ordinary (linear) pricing together with quantity constraints. That is, the monopolist chooses both the price of a good and a maximum

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quantity he is willing to sell at the price. This kind of price schedule is interesting not only because it is simple, but also because it simulates the kind of pricing embodied in many tariff-quota schemes. It also gives rise to the type of quantity rationing found in the fixed price literature.

In this paper we examine when a monopolist would use such a price schedule. More specifically, we consider a price-setting agent (the "monopolist") constrained to choose linear prices and ask if he would find setting quantity rations desirable as well. We show first (Theorem 1) that if all buyers are identical, quantity rations are not useful for the monopolist. This demonstrates that the first kind of price discrimination mentioned above cannot be practiced using linear prices and quantity constraints alone. However, we show by means of Example 1 that quantity constraints may be desirable with a diversity of buyers, illustrating the possibility of price discrimination of the second kind. In this example, the monopolist sets a *zero* constraint for one good, which prevents competitive agents from buying that good at all. We assert, moreover (Theorem 2), that in a model of only two goods, a monopolist will use only this kind of quantity constraint. That this result does not generalize to more than two goods is established in Example 2, which illustrates that nonzero quantity constraining may be desirable with sufficiently diverse buyers in models of at least three goods.

The approach in this paper should be distinguished from that in the recent work of Guesnerie and Roberts [1980]. In our analysis we do not permit agents to be constrained to trade *more* than they would like of any good; whereas in their treatment of taxation with quantity constraints, Guesnerie and Roberts make no such prohibition.

THE MODEL

We consider an exchange economy consisting of $m + 1$ agents, indexed by $i = 0, \ldots, m$, and l commodities, indexed by $h = 1, \ldots, l$. Agent i has an endowment $w^i \epsilon R^l_+$ and a utility function $u^i: R^l_+ \rightarrow R$. We shall take agent 0 to be the "monopolist." He chooses price vector p in the unit simplex and quantity constraint vectors $\underline{z} \epsilon R^l_-$ [1] and $\bar{z} \epsilon R^l_+$ to maximize his utility given the price- and quantity-taking

1. R^l_+ denotes the nonnegative l-dimensional orthant; R^l_- denotes the nonpositive l-dimensional orthant.

responses of the other (competitive) agents. That is, he chooses $p, \underline{z}^i$, and $\bar{z}^i$ [2] to maximize

$$u^0\left(-\sum_{i=1}^{n} z^i + w^0\right),$$

where z^i maximizes $u^i(z^i + w^i)$ subject to $pz^i = 0$ and $\underline{z}^i \leq z^i \leq \bar{z}^i$. If either of these last inequalities is binding in a solution to this program, the solution is said to involve *rationing.*

IDENTICAL AGENTS

We first show that, in the case of a single competitive agent or, equivalently, multiple but identical agents, quantity constraints do not help a monopolist (at least, if demand is sufficiently well-behaved). This proposition is obvious for a model with only two goods if the monopolist's preferences are monotonic. If an agent is constrained in buying good 1, the monopolist can raise p_1 slightly without affecting the agent's demand for good 1 but increasing the (agent's) sale of good 2 (assuming that the agent's demand varies continuously in price); this must represent a gain for the monopolist. It may appear, intuitively, that this argument does not extend to more than two goods. Could it not be that as the monopolist raises the price of a demand-constrained good, the demand and supply for other goods changes unfavorably for him? Theorem 1 answers this question in the negative.

THEOREM 1. In the case $m = 1$, suppose that the monopolist's preferences are monotonic,[3] and that the competitor's preferences are continuous, monotonic, and strictly convex.[4] Then there is a solution to the monopolist's program that involves no rationing.

The proof will be carried out in three steps. Throughout we shall maintain the hypotheses of Theorem 1. Consider the offer surface of the single competitor:

2. In this formulation we allow the monopolist to assign different agents different rations. This is in part to accommodate the variety of rationing schemes considered in the fixed price literature. However, no result in this paper would be changed if we required the monopolist to set *uniform* rations; i.e., $\bar{z}^i = \bar{z}$ and $\underline{z}^i = \underline{z}$ for all i.

3. By "monotonicity" we mean that more of any good is strictly preferred.

4. Strict convexity, rather than weak convexity, is inessential but somewhat simplifies the argument.

$$Z = \{z \in R^l | \quad \exists p \neq 0, pz = 0, \text{ and } pz' = 0 \text{ implies that } u^1(z + w^1) \geqq u^1(z' + w^1)\}.$$

Let

$$\text{Star } Z = \{z \in R^l | \quad z = \alpha z' \text{ for } z \in Z \text{ and } 0 \leq \alpha \leq 1\}$$

denote the star-shaped hull of Z. We first show that if a competitor is constrained by $(\underline{z},\bar{z})$, then his constrained demand, given prices, lies in Star Z if it does not lie in R^l_+.

LEMMA 1. Let $\hat{z}$ be a solution for

$$\max\{u^1(z + w^1) | \quad \hat{p}z = 0, \underline{z} \leqq z \leqq \bar{z}\}$$

at $\hat{p}$. Then if $\hat{z} \notin R^l_+, \hat{z} \in \textit{Star } Z$.

Proof.[5] We first demonstrate that for any $z \notin R^l_+$ such that $u^1(z + w^1) > u^1(w^1)$, there exists $\lambda \neq 0$ such that $\lambda z \in Z$. Consider the line segment $L(z) = \{\tilde{z} \in R | \tilde{z} = \lambda z, \lambda \in R, \tilde{z} + w^1 \geq 0\}$. Because $z \notin R^l_+$, it has at least one negative component. Hence, $L(z)$ is compact, and there exists a maximizer z' for u^1 on $L(z)$. Writing $z' = \lambda' z$, we conclude that $\lambda' \neq 0$, since $u^1(z + w^1) > u^1(w^1)$. Convexity of the competitor's preferences implies that there exist prices $p' \neq 0$ such that if $u^1(\tilde{z} + w^1) > u^1(z' + w^1)$, then $p'\tilde{z} > p'z'$; and if $\tilde{z} = \lambda z'$ for some $\lambda \in R$, then $p'z' \geq p'\tilde{z}$. But this last implication means that $p'z' = 0$. Therefore, $z' \in Z$.

Next consider the competitor's optimal trade $\hat{z}$ given prices $\hat{p}$ and constraints $(\underline{z},\bar{z})$. Since by hypothesis $\hat{z} \notin R^l_+, \hat{z} \neq 0$. Therefore, from strict convexity, $u^1(\hat{z} + w^1) > u^1(w^1)$. Thus, by the argument of the preceding paragraph, there exists $\hat{\lambda} \neq 0$ such that $\hat{\lambda}\hat{z} \in Z$. If $\hat{\lambda} < 0$, then, since $u^1(\hat{\lambda}\hat{z} + w^1)$, strict convexity implies that $u^1(w^1) > u^1(w^1)$, an impossibility. If $0 < \hat{\lambda} < 1$, then $\underline{z} \leqq \hat{\lambda}\hat{z} \leqq \bar{z}$. But since $u^1(\hat{\lambda}\hat{z} + w^1) \geq u^1(\hat{z} + w^1)$ and $\hat{z}$ is optimal given $\hat{p}$ and $(\underline{z},\bar{z})$, $\hat{\lambda}\hat{z}$ must then also be optimal, a violation of strict convexity. Hence, $\lambda \geq 1$, and so $\hat{z} \in$ Star Z.

Q.E.D.

Let Δ^{l-1} be the l-dimensional simplex and $z \in R^l$ such that $z + w^1 \geq 0$. Let $P(z) = \{p \in \Delta^{l-1} | \quad u^1(y) \geq u^1(z + w^1) \text{ imply that } py \geq p(z + w^1)\}$, and define $V(z) = P(z)z$. Since the competitor's preferences are convex and continuous, $V(z)$ is a convex-valued, upper hemicontinuous correspondence. Note that if $0 \in V(z)$, then $z \in Z$.

5. We are grateful to H. Sonnenschein for this line of argument.

LEMMA 2. If $z \in \text{Star } Z\backslash Z$, then $V(z) > 0$.

Proof. Suppose not. Then there exists $p \in P(z)$ with $pz \leq 0$. If $pz = 0$, then $z \in Z$, a contradiction. Thus, $pz < 0$. Since $z \in \text{Star } Z\backslash Z$, there exists $\lambda > 1$ such that $\lambda z \in Z$. Then $pz > p(\lambda z)$, and so $u^1(z + w^1) \geq u^1(\lambda z + w^1)$. But since $\lambda z \in Z$, there exists $\tilde{p} \in P(\lambda z)$ such that $\tilde{p}(\lambda z) = 0$. Hence $\tilde{p}z = 0$, and so $z \in Z$, a contradiction.

Q.E.D.

Proof of Theorem 1. Let $\hat{z}$ be the competitor's net trade in a solution to the monopolist's program. If $\hat{z} \in Z$, we are done. Therefore, assume that $\hat{z} \notin Z$. Then $\hat{z} \neq 0$, and so, from monotonicity of the monopolist's preferences, $\hat{z} \notin R^l_+$. From Lemma 1, $\hat{z} \in \text{Star } Z\backslash Z$. From Lemma 2 $V(\hat{z}) > 0$. For $\epsilon \geq 0$ define z^ϵ so that

$$z^\epsilon_h = \begin{cases} \hat{z}_h, & \text{if } \hat{z}_h \leq 0 \\ \max(0, \hat{z}_h - \epsilon), & \text{otherwise} \end{cases}$$

$$h = 1, \ldots, l.$$

Because the competitor's preferences are monotonic, for all ϵ there exists $p \in P(z^\epsilon)$ such that $p_h > 0$ for all h. Thus, for ϵ sufficiently large, $V(z^\epsilon) < 0$. By the convex-valuedness and upper hemicontinuity of V, $V(z^{\bar{\epsilon}}) = 0$ for some $\bar{\epsilon} > 0$. Then $z^{\bar{\epsilon}} \in Z$. By the monotonicity of the monopolist's preferences, $u^0(w^0 - z^\epsilon) > u^0(w^0 - \hat{z})$, a contradiction of $\hat{z}$'s optimality.

Q.E.D.

We have established Theorem 1 for the case of a single (or several identical) competitive agent(s). It is evident, however, that the same proof goes through if agents are diverse but the monopolist can assign different agents different prices.

The continuity and convexity assumptions in Theorem 1 are invoked to ensure that the competitor's demand is continuous. In Figure I the monopolist finds it advantageous to constrain the competitor because of the discontinuity in demand represented by the dotted line. Heal [1981] develops this line by linking the use of quantity constraints to nonconvexities in production.

DIVERSE AGENTS

We next turn to the case of diverse agents. Here it is straightforward to give examples in which quantity constraints help the monopolist.

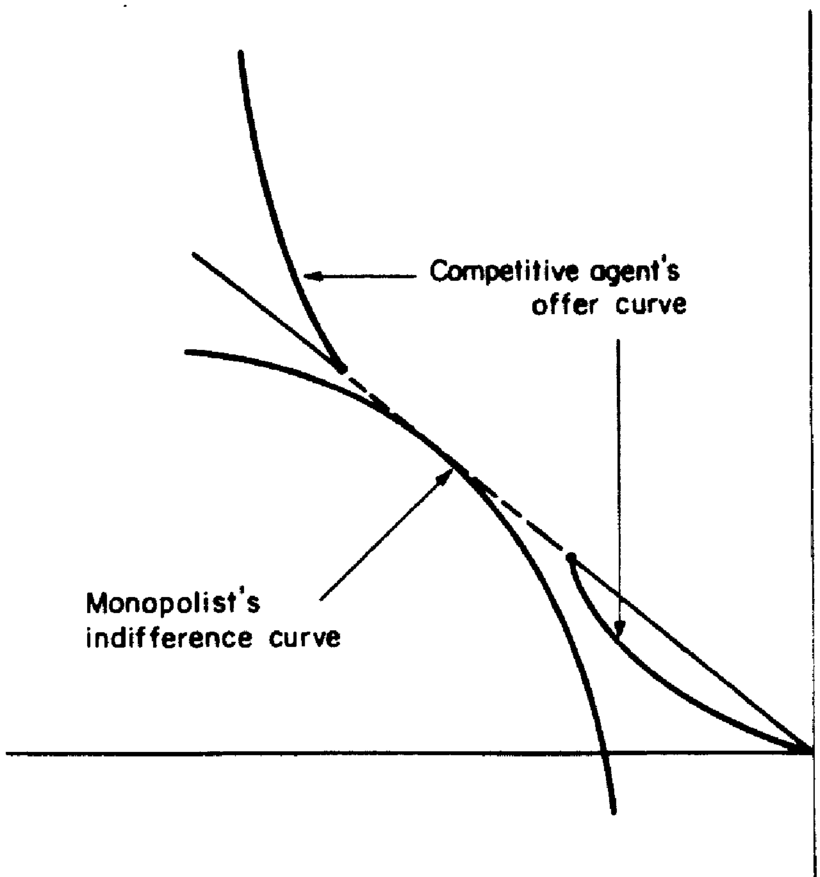

FIGURE I

Example 1

$$m = 2, \quad l = 2$$

$$u^1(x_1,x_2) = u^2(x_1,x_2) = x_1^{1/2}x_2^{1/2}$$

$$w^1 = (1,0), \quad w^2 = (0,1).$$

Take $u^0(x_1,x_2) = 2x_1 + x_2$. The joint offer curve of agents 1 and 2 is described by the curve $x_2 = -\frac{1}{2} + 1/(4x_1 + 2)$. Maximizing $2(-x_1) - x_2$ along this curve, the monopolist attains a maximum payoff of $(3 - 2\sqrt{2})/2$. The offer curve of agent 1 is described by the curve $x_1 = -\frac{1}{2}$ for $x_2 > \frac{1}{2}$. Maximizing along this curve, the monopolist can reach a payoff arbitrarily close to 1. Because $1 > (3 - 2\sqrt{2})/2$, the monopolist does better to prevent agent 2 from trading at all than to leave him unconstrained.

In Example 1, quantity constraints emerge in an extreme way: One agent is constrained to zero trade. The example is nonetheless entirely representative of the kind of quantity constraints that arise in models with only two goods.

THEOREM 2. If $l = 2$, competitive agents' preferences are convex and continuous, and the monopolist's preferences are weakly monotonic,[6] then there exists a solution to the monopolist's program in which, for all h and i, $\bar{z}^i_h$ and $\underline{z}^i_h$ are either infinite or zero.

Proof. Suppose that the hypotheses of the theorem are satisfied. Let z^* be an aggregate net trade that corresponds to the monopolist's optimal choice of prices and constraints $(p^*,\{\bar{z}^{*j}\},\{\underline{z}^{*j}\})$. (If there is more than one such z^*, the monopolist chooses the optimal one from his own standpoint.) Without less of generality, we may assume that $z^*_2 \geq 0$. If $z^*_2 = 0$, then $z^*_1 = 0$, and the monopolist can induce this aggregate net trade by taking $\bar{z}^{*j} = \underline{z}^{*j} = 0$ for all j. Therefore, assume that $z^*_2 > 0$.

Suppose first that there is an agent i who is constrained in his demand for good 2 (equivalently, his supply of good 1). That is, suppose that $\bar{z}^{*i}_2$ is a binding constraint. Suppose that the monopolist raises the price of good 2 slightly above p^*_2 ($p^*_2 < 1$ because $p^*_1 z^*_1 + p^*_2 z^*_2 = 0$ and $z^*_2 > 0$). If aggregate demand for good 2 rises given $(\{\bar{z}^{*j}\},\{\underline{z}^{*j}\})$, the monopolist can tighten the constraints $\bar{z}^i_2$ both to reduce demand for good 2 and to increase sales of good 1 compared with z^*_2 and z^*_1. Given the monopolist's weakly monotonic preferences, he is then better off, contradicting the optimality of $(p^*,\{\bar{z}^{*j}\},\{\underline{z}^{*j}\})$. Therefore, assume that aggregate demand falls with the price rise. If demand falls continuously,[7] the monopolist can relax the constraint on agent i, again both increasing the aggregate sale of good 2 and decreasing the aggregate purchase of good 1. If demand falls discontinuously, then given the assumptions of convexity and continuity, the discontinuity must be due to the indifference by some agents among alternative net trades. That is, given $(p^*,\{\bar{z}^{*j}\},\{\underline{z}^{*j}\})$, there exists a set of competitive agents J such that each agent $j \in J$ is indifferent among a line segment $[\hat{z}^j,\hat{z}^j]$ of net trades containing z^{*j} with $\hat{z}^j_2 < z^{*j}_2$.
For $\lambda \in [0,1]$ and $j \in J$ take

$$z^j(\lambda) = \lambda\hat{z}^j + (1 - \lambda)z^{*j}$$

$$\bar{z}^j(\lambda) = \bar{z}^{*j}$$

$$\underline{z}^j(\lambda) = \underline{z}^{*j}.$$

6. By "weak monotonicity" we mean that if a consumption bundle is greater in every component than another, then it is strictly preferred.

7. Because we assume only convexity, rather than strict convexity, the demand correspondence need not be continuous (merely upper hemicontinuous). Thus, starting from some net trades in the demand correspondence, a change in price may necessarily imply a discontinuous change in demand.

For $j \notin J$, and if they exist, choose

$$\bar{z}^j(\lambda) \geq \bar{z}^{*j}$$

$$\underline{z}(\lambda) \leq \underline{z}^{*j}$$

so that, for some choice of optimal trades $\{z^j(\lambda)\}_{j \notin J}$ given $(p^*, \bar{z}^j(\lambda), \underline{z}^j(\lambda))$, $\Sigma_{j=1}^{n} \bar{z}^j(\lambda) = z^*$. Because $\hat{z}_2^i < z_2^{*j}$ and agent i is constrained in his demand for good 2 at $(p^*, \bar{z}^i, \underline{z}^i)$, such a choice of $\{(\bar{z}^j(\lambda), \underline{z}^j(\lambda))\}_{j \notin J}$ can be made for all $\lambda \leq \lambda^*$, where $\lambda^* > 0$ is the maximum value for which there is such a solution. If $\lambda^* = 1$, then the change in $z_2(1)$ $(= \Sigma\, z_2^i(1))$ is continuous when p_2 increases slightly from p_2^* and we may apply the preceding argument. If $\lambda^* < 1$, then $\bar{z}_2^i(\lambda^*)$ cannot be binding for any j, since, if it were, we could relax it and further increase λ. Thus, we conclude that there exists a solution to the monopolist's problem in which no agent's demand for good 2 is constrained.

Suppose then that there exists an agent i who is constrained in his sale of good 2 and that no agent is constrained in his demand for good 2. Consider increasing p_2 slightly from p_2^*. If aggregate demand for good 2 rises, the monopolist can tighten the constraint on this demand to obtain the same contradiction as before. If aggregate demand falls discontinuously, then, as in the preceding paragraph, there exists another solution to the monopolist's program in which either aggregate demand changes continuously or no agent is constrained. We need consider, therefore, only continuous decreases in demand. But in this case, the monopolist can tighten the constraint on agent i so as to increase sales of good 1 above $|z_1^*|$ and decrease demand for good 2 below z_2^* unless agent i is already constrained to zero. Thus, if $z_2^* > 0$, the only constraint that can be optimal for the monopolist is to constrain the sale of good 2 to zero.

Q.E.D.

Theorem 2 does not extend to more than two goods as the following example illustrates.

Example 2

$m = 2$, $l = 3$. Let $w^0 = (0,2,1)$, $w^1 = (2,0,0)$, $w^2 = (1,0,0)$

and take

$$u^0(x_1,x_2,x_3) = x_1$$

$$u^1(x_1,x_2,x_3) = x_1 + 2x_2 + x_3$$

$$u^2(x_1,x_2,x_3) = x_1 + x_2.$$

Let

$$p_1 = 1/3.$$

(i) If $p_2 < 1/3$, then aggregate demand for commodity 2 is greater than two or that for commodity 3 is greater than one.

(ii) Let $p_2 = 1/3$ and $1/6 < p_3 < 1/3$. Then the utility-maximizing net trades for the two agents are $z^1 = (-2,2,0)$ and $z^2 = (-\alpha,+\alpha,0), 0 \leq \alpha \leq 1$. Any such prices are market-clearing, with $\alpha = 0$ generating a utility of two for the monopolist.

(iii) Let $p_2 = 1/3$ and $p_3 = 1/6$. Then $z^1 = (-2,2-(\beta/2),\beta), 0 \leq \beta \leq 4$ and $z^2 = (-\alpha, \alpha, 0), 0 \leq \alpha \leq 1$. Feasibility requires $\beta \leq 1$ and $2 - (\beta/2) + \alpha = 2$, which yields a maximal $\alpha = 1/2$ and a utility for the monopolist of $5/2$.

(iv) If $p_2 > 1/3$, then the utility of the monopolist can never exceed two.

Therefore, (iii) is the optimum for the monopolist without rationing. However, with prices $p = (1/3,1/3,1/3)$ and a demand constraint of one unit of commodity 2 per customer, the monopolist induces net trades $z^1 = (-2,1,1)$ and $z^2 = (-1,1,0)$ and a utility of 3.

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