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Two Examples of Equilibria Under Price Rigidities and Quantity Rationing

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1. Introduction

In a recent paper Drèze (1975) proposed an equilibrium concept for an exchange economy where prices are not sufficiently flexible to achieve a Walrasian equilibrium. In such a case individual utility maximization guided only by price signals will not clear all markets. The concept of a quantity rationing process is introduced to impose additional restrictions on individual trades such that markets will clear. In this situation each individual receives price as well as quantity signals and maximizes his utility under these joint constrainst.

Two basic properties of the quantity rationing at equilibrium are imposed. First, the quantity constraints on net trades for each individual in each market are independent of his own actions and they determine only upper bounds for his net trades. In such a case an individual cannot manipulate these constraints in his favor by e. g. overstating his demand or supply, and he is never forced to trade since he can always propose less than the constraint, in particular zero. Second, the quantity constraints are effective on one side of each market only, i. e. if for some market some agent is rationed on the demand (supply) side, then no agent perceives constraints on the supply (demand) side of this market, which are binding for his equilibrium decision.

It is well known, that the set of such equilibria for a given fixed price system is very large in general. If no other restrictions are imposed, like e. g. that not all markets can be rationed, there always exists an equilibrium with zero trade, if all prices are positive. All this makes clear, that an equilibrium with quantity rationing may be very inefficient. On the other hand it is immediately apparent that, in general, global Pareto optimality cannot be achieved as long as prices are not Walrasian and income transfers are not allowed. The appropriate criterion in this case then has to be one of constrained Pareto optimality given that all agents trade at the given prices on their budget sets. Recently Younès (1975) indicated in a slightly different context that in general an equilibrium with quantity rationing will not be a constrained Pareto optimum. The two examples in this paper give a general geometric illustration of this fact using an appropriately modified representation in the conventional Edgeworth box diagram. The examples also suggest, that constrained Pareto optimality may be obtained, if the rationing mechanism assures some interdependence between markets.

2. Definitions

Let $\mathscr{E}=\{I, (X_i, \omega_i, \underset{i}{\geqslant})\}$ denote an exchange economy with a finite set of consumers $I=\{1,\ldots, i,\ldots, n\}$. There are l commodities indexed $h=1,\ldots, l$. Then with the usual interpretation R^l is the commodity space, $X_i\subset R^l$ is consumer i's consumption set, $\omega_i \in R^l_+$ denotes consumer i's initial endowment of commodities and $\underset{i}{\succsim}$ denotes his preference ordering on X_i.

Given a fixed price vector $p \in R^l$, $p\neq 0$, consumer i is constrained to bundles $x_i \in R^l$ whose value does not exceed the value of his initial endowment. If he desires x_i his net trade is defined as $z_i=x_i-\omega_i$. The rationing of each consumer i is described by a pair $(\underline{z}_i, \bar{z}_i) \in R^l_-\times R^l_+$ where $\underline{z}_i^h$, $h=1,\ldots, l$ represents the maximum quantity of commodity h which consumer i is allowed to sell and where $\bar{z}_i^h$, $h=1,\ldots, l$ represents the maximum quantity of commodity h which he is allowed to purchase. Given prices p and the rationing constraints $(\underline{z}_i, \bar{z}_i)$, consumer i chooses a net trade z_i which maximizes his preference relation subject to his budget constraint and the rationing constraints.

Let

$$\beta_i (p, \underline{z}_i, \bar{z}_i)=\{z_i \in R^l | p\cdot z_i=0, \underline{z}_i\leq z_i\leq \bar{z}_i, \omega_i+z_i \in X_i\}$$

denote the constrained budget correspondence of consumer i. His demand correspondence γ_i is then defined by

$$\gamma_i (p, \underline{z}_i, \bar{z}_i)=\{z_i \in \beta_i (p, \underline{z}_i, \bar{z}_i) | \omega_i+z_i \underset{i}{\succsim} \omega_i+z_i' \text{ for all } z_i' \in \beta_i (p, \underline{z}_i, \bar{z}_i)\}.$$

Definition 1 (according to Drèze (1975)): An equilibrium with quantity rationing at prices $p \in R^l$, $p \neq 0$, is a list of net trades (z_i), $i=1,\dots,n$, and a list of constraints $(\underline{z}_i, \bar{z}_i)$, $i=1,\dots,n$ such that

(1) $\sum_{i=1}^{n} z_i = 0$

(2) $z_i \in \gamma_i(p, \underline{z}_i, \bar{z}_i)$ for all $i \in I$

(3) for all $h=1,\dots,l$

(i) $z_j^h = \underline{z}_j^h$ some $j \in I$ implies $\bar{z}_i^h > z_i^h$ all $i \in I$

(ii) $z_j^h = \bar{z}_j^h$ some $j \in I$ implies $z_i^h > \underline{z}_i^h$ all $i \in I$

Conditions (1) and (2) need no comment. Condition (3) reflects the fact that on each market h at most one side is effectively rationed.

Definition 2: A list of net trades (z_i), $i=1,\dots,n$, is a constrained Pareto optimum at prices p if

(1) $\sum_{i=1}^{n} z_i = 0$

(2) $p \cdot z_i = 0$ for all $i \in I$

(3) there exists no other list of net trades (z_i'), $i=1,\dots,n$, such that

(i) $\sum_{i=1}^{n} z_i' = 0$

(ii) $p \cdot z_i' = 0$ for all $i \in I$

(iii) $z_i' + \omega_i \underset{i}{\gtrsim} z_i + \omega_i$ for all $i \in I$

with strict preference for at least one $i \in I$.

Without causing confusion of terminology any final allocation (x_i), $i=1,\dots,n$, associated with an equilibrium net trade (z_i), where $x_i = z_i + \omega_i$, will also be called an equilibrium with quantity constraints. Similarly, if (z_i), $i=1,\dots,n$, is a constrained Pareto optimum, then (x_i), $i=1,\dots,n$, defined by $x_i = \omega_i + z_i$, will also be called a constrained Pareto optimum. The following two examples demonstrate that the two sets of allocations in general have an empty intersection. The first example indicates the general characteristics of the two sets. The second more specific example shows that a continuous change of the constraints may result in a continuous increase in utility from the zero trade equilibrium to the best equilibrium which is, however, at a positive distance from the set of constrained Pareto optima. Furthermore it is shown that the zero trade situation may be the only equilibrium if some prior restrictions are imposed on which markets should be excluded from the rationing mechanism.

3. Two Examples

Consider an economy with two consumers i and j and three commodities $h=1, 2, 3$. Let $X_i = X_j = R^3_+$ and assume that consumer i has nothing of the second commodity and consumer j has nothing of the first commodity, but both own positive amounts of the other two commodities initially. Then $\omega^i = (\omega_1{}^i, 0, \omega_3{}^i)$ and $\omega^j = (0, \omega_2{}^j, \omega_3{}^j)$. Finally, preferences for both consumers are assumed to be strictly convex and monotonic.

Given the vector of fixed prices $p \gg 0$, $p \cdot x = p \cdot \omega$, $x \in R^3_+$, defines the budget plane for each consumer and for any choice (x_1, x_2) of the first two commodities the quantity of the third commodity is uniquely defined. Therefore the allocation problem can be described completely by an analysis of a two dimensional Edgeworth box with some straightforward modifications.

Consider consumer i. His consumption possibilities of the first two commodities restricted to his budget plane are given by the projection of his budget plane into the plane $R^2 \times \{0\}$ (see Fig. 1). The set of feasible (x_1, x_2)-consumption plans is described by the triangle $0^i A^i B^i$. His unconstrained optimal consumption plan is the point $(\bar{x}_1{}^i, \bar{x}_2{}^i)$ which has been chosen to be on the line $p_1 x_1{}^i + p_2 x_2{}^i = p_1 \omega_1{}^i$. This implies that his unconstrained net trade on the third market is equal to zero. His demand on market two is equal to $\bar{x}_2{}^i$ and his supply on market one is equal to $\omega_1{}^i - \bar{x}_1{}^i$. His indifference curves in the triangle $0^i A^i B^i$ take the form of closed curves around the point $(\bar{x}_1{}^i, \bar{x}_2{}^i)$.

If consumer i is rationed on market one, then, for alternative values of the constraint between $\omega_1{}^i - \bar{x}_1{}^i$ and zero one obtains his constrained offer curve $S_1{}^i$. In Fig. 1 $S_1{}^i$ has been drawn to imply a simultaneous reduction of his demand of commodity two and of his supply of commodity three. Similarly, for alternative rationing levels on market two, one obtains his constrained offer curve $D_2{}^i$. The no forced trade assumption implies that rationing on market three alone will not affect consumer i's optimal decision. On the other hand it can be seen easily from Fig. 1 that any rationing on more than one market will make him choose a best consumption plan between the two curves $S_1{}^i$ and $D_2{}^i$.

Fig. 2 shows the same analysis for consumer j. There, however, it has been assumed that his unconstrained optimal consumption plan involves a positive demand on market three. Consumer j's unconstrained supply on market two is equal to $\omega_2{}^j - \bar{x}_2{}^j$ and his demand on market one is equal to $\bar{x}_1{}^j$. Rationing on market one

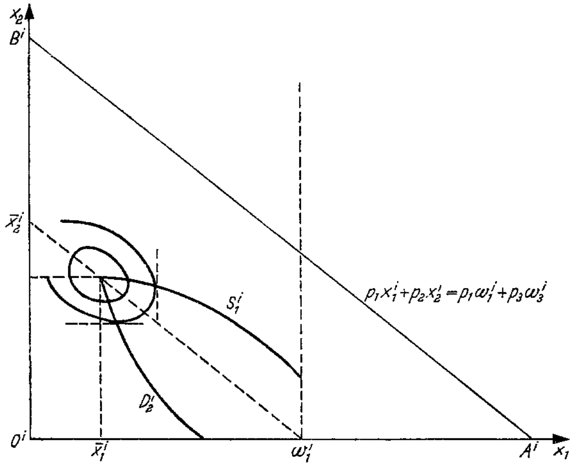

Fig. 1

only yields the constrained offer curve $D_1{}^j$. Similarly, for market two one obtains $S_2{}^j$ and for market three $D_3{}^j$. Rationing on more

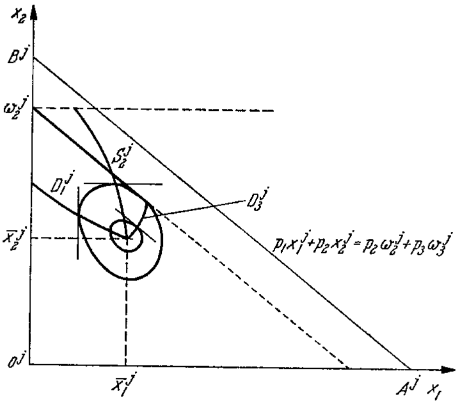

Fig. 2

than one market simultaneously results in a decision which can be written as a convex combination of the endowment and some point on one of the three curves.

Fig. 3 gives the complete diagram. The zero consumption points of i and j are placed in the usual way. The joint endowment point is A. M^i and M^j denote the unconstrained optimal consumption plans of i and j respectively. It is easily seen that E_1, E_2, and E_3 are the only equilibria which require rationing on two markets only. E_1 is a pure supply constrained equilibrium with rationing on markets one and two. E_2 represents an equilibrium with rationed supply on market one and rationed demand on market three. At E_3 both consumers are rationed at a zero demand level on market three and consumer i is rationed on market two as well. At E_0 consumer j is rationed on market two whereas consumer i is rationed on

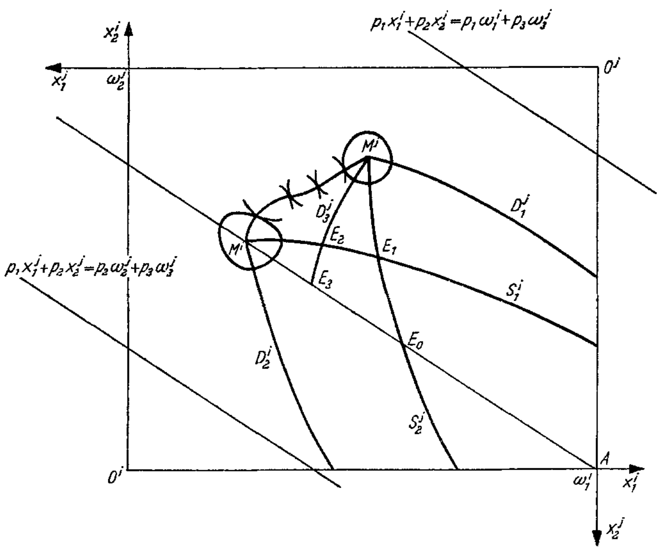

Fig. 3

markets one and three. It is now easy to verify that the four line segments E_0E_1, E_1E_2, E_2E_3, and E_3A constitute the set of equilibria with quantity rationing. On the other hand the set of constrained Pareto optima is the contract curve connecting M^i with M^j which is disjoint from the set of equilibria.

The purpose of the second example is to display some additional features of the relationship between the set of constrained Pareto optima and the set of equilibria. For the same economy it will be assumed that the preferences of both consumers are identical and representable by the utility function $u(x_1, x_2, x_3) = 2x_1x_2 + x_2x_3 + x_3x_1$. Endowments in this case will be $\omega^i = (1, 5, 4)$ and $\omega^j = (5, 4, 1)$.

Prices are all identical and equal to unity. Instead of the previous geometric representation in a normal Edgeworth box we will use an equivalent one here in the space of net trades for commodities one and two.

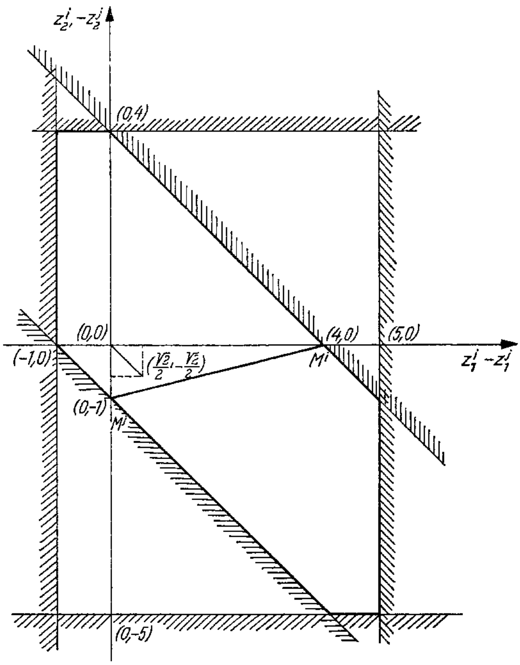

Fig. 4

The set of feasible net trades for a consumer is defined by

$$Z = \{z \in R^3 | \omega + z \geq 0, z_1 + z_2 + z_3 = 0\}.$$

Eliminating z_3 one obtains as the feasible sets in $z_1 - z_2$-space for i and for j

$$Z^i = \{(z_1, z_2) | z_1 + 1 \geq 0, z_2 + 5 \geq 0, 4 - z_1 - z_2 \geq 0\}$$

$$Z^j = \{(z_1, z_2) | z_1 + 5 \geq 0, z_2 + 4 \geq 0, 1 - z_1 - z_2 \geq 0\}$$

Eliminating z_3 as an argument of the utility function one obtains

$$\tilde{u}^i (z_1, z_2) = -(z_1 - 4)^2 - z_2{}^2 + c^i$$

$$\tilde{u}^j (z_1, z_2) = -z_1{}^2 - (z_2 + 1)^2 + c^j,$$

where c^i and c^j are constants. Fig. 4 gives the geometric representation in the z_1-z_2-plane with z^i for consumer i and $-z^j$ for consumer j. Consumer i's unconstrained net trade consists of a demand of four units of commodity one against four units of commodity three offered for sale, so that M^i is the point (4, 0). Consumer j maximizes his utility at a positive demand of one unit of commodity two and a sale of one unit of commodity three. Hence his maximal point is $-z^j=(0, -1)$. The straight line joining M^i with M^j is the set of constrained Pareto optima.

There are two equilibria which involve rationing on two markets only. For consumer i, his demand constrained (net) offer curve with respect to rationing on market one coincides with the line segment [(0, 0), (4, 0)] since the indifference curves are concentric circles around M^i. For consumer j, the same argument shows that the segment [(0, 0), (0, −1)] is his demand constrained offer curve with respect to market one. The intersection of the two curves at the no trade position is an equilibrium where demand is rationed on market one and on market two at a zero level. The other equilibrium is obtained for a zero rationing of sales for both consumers on market three and a demand rationing level of $\frac{1}{2}\sqrt{2}$ on market one for consumer i. The zero rationing on market three imposes on both consumers to trade along the straight line of slope minus one which passes through the origin. Therefore consumer i supplies $\frac{1}{2}\sqrt{2}$ units of commodity two. Some straightforward arguments show that $-(\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2})$ is the optimal net trade for consumer j if he is rationed at a zero level on market three. Finally, it is easy to verify that the whole line segment $[(0, 0), (\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2})]$ is the set of all equilibria with quantity rationing. However, all intermediate points involve rationing on all three markets.

Again it is clear from Fig. 4 that the set of equilibria and the set of constrained Pareto optima are disjoint. The optimality properties of the set of equilibria, however, are strongly dependent on which and on how many markets are to be rationed. If, for example, at least one market should not be rationed then, by determining this market a priori this may result in a zero trade situation as the only equilibrium.

It is clear that successive increases of the rationing levels from zero to $(\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2})$ result in a continuous increase in utility for both consumers. However, further increases are not possible as is shown by the following arguments.

Given any Pareto preferred points (z_1, z_2) with respect to $(\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2})$, Fig. 5 shows that $z_1>\frac{1}{2}\sqrt{2}$ and $-z_2>\frac{1}{2}\sqrt{2}$.

However there exist $\lambda \in (0, 1)$ such that consumer j prefers $\lambda(-z_1, -z_2)$ over $(-z_1, -z_2)$. Hence (z_1, z_2) cannot be substained

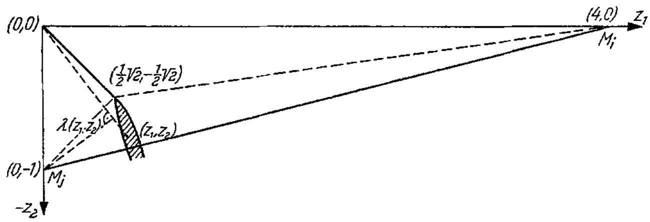

Fig. 5

as an equilibrium. Therefore, a Pareto improvement requires some form of forced trading.

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