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ON THE CONTINUITY OF THE OPTIMAL POLICY SET FOR LINEAR PROGRAMS*

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Abstract. For a linear program it is shown that the optimal policy set behaves continuously if the constraint vector changes on the set for which the program has a solution. The result implies that there exists a continuous optimal policy function for which a construction is indicated.

1. Introduction. Consider the linear program

$$P: \text{Minimize } c \cdot x$$

subject to

$$Ax \geqq b, \quad x \geqq 0,$$

where $c, x \in R^l$, $b \in R^m$ and A is an $m \times l$ matrix. For fixed A and c define

$$\Pi(b) = \min \{c \cdot x | Ax \geqq b, x \geqq 0\},$$

$$\zeta(b) = \{x \in R^l | Ax \geqq b, x \geqq 0, c \cdot x = \Pi(b)\}.$$

Let $B \subset R^m$ be the set of $b \in R^m$ for which P has a solution. If B is nonempty, then it is equivalent to take B as the set of $b \in R^m$ for which P is consistent, i.e., B is the set of "feasible right-hand sides". Π is a function from B to the real numbers and ζ is a set-valued mapping from B to R^l. The purpose of this note is to show a strong continuity property of ζ which implies the existence of a continuous selection function.

General continuity properties of optimal policy sets were discussed in different contexts in [3], [4] and in [5]. For the linear case, Theorem 2 of this paper supplies a stronger continuity result than the results in [4] and in [5] without requiring compactness of ζ.

2. Preliminaries. A set-valued function τ of a set $X \subset R^l$ into the set of subsets of $Y \subset R^m$, which assigns to each $x \in X$ a subset $\tau(x) \subset Y$, is called a correspondence if for every $x \in X$, $\tau(x) \neq \emptyset$. Let 2^Y denote the set of subsets of Y.

DEFINITION 1. The correspondence τ from X into 2^Y is called *upper hemicontinuous* (u.h.c.) at $x \in X$ if for every open neighborhood U of $\tau(x)$ there exists an open neighborhood V of x such that for every $z \in V$, $\tau(z) \subset U$.

DEFINITION 2. The correspondence τ from X into 2^Y is called *lower hemicontinuous* (l.h.c.) at $x \in X$ if for every open set M in Y such that $M \cap \tau(x) \neq \emptyset$, there exists an open neighborhood V of x such that for all $z \in V$, $M \cap \tau(z) \neq \emptyset$.

DEFINITION 3. The correspondence τ from X into 2^Y is called *continuous at* x if it is lower and upper hemicontinuous at x.

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DEFINITION 4. The correspondence τ from X into 2^Y is called *lower hemicontinuous, upper hemicontinuous,* or *continuous* if it is lower hemicontinuous, upper hemicontinuous, or continuous respectively at every $x \in X$.

Clearly, if τ is single-valued and u.h.c. or l.h.c., then τ is a continuous function. The following propositions are stated without proofs. An extensive description of the continuity properties of correspondences can be found in W. Hildenbrand [6].

PROPOSITION 1. *Let ϕ and ψ be correspondences of X into 2^Y and assume that $\phi(x) \cap \psi(x) \neq \emptyset$ for all $x \in X$. If ϕ has a closed graph and if ψ is u.h.c. and compact-valued, then the correspondence τ of X into 2^Y defined by $\tau(x) = \phi(x) \cap \psi(x)$ is u.h.c.*

PROPOSITION 2. *Let τ be a correspondence from $X \subset R^l$ into 2^Y. τ is l.h.c. if and only if for every sequence $(x^n)_{n=1,\cdots}$ in X converging to $x \in X$ and $y \in \tau(x)$ there exists a sequence $(y^n)_{n=1,\cdots}$ converging to y such that $y^n \in \tau(x^n)$ for all $n = 1, \cdots$.*

3. Main results.

THEOREM 1. *$\Pi : B \to R$ is continuous.*

Theorem 1 is well known and a proof will not be given (see, for example, [8]).

THEOREM 2. *The correspondence ζ from B into subsets of R^l has a closed graph and is lower hemicontinuous. ζ is continuous if $\zeta(b)$ is compact for some $b \in B$.*

Proof. Clearly, $\zeta(b) = \{x \in R^l | \hat{A}x \geqq \hat{b},\ c \cdot x \leqq \Pi(b)\}$, where $\hat{A} = \begin{pmatrix} A \\ I \end{pmatrix}$ with I being the $l \times l$ identity matrix and

$$\hat{b} = \begin{pmatrix} b \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}.$$

$\zeta(b)$ has a closed graph, since for every sequence (b^n, x^n) converging to (b, x), $\hat{A}x^n \geqq \hat{b}^n$ and $c \cdot x^n \leqq \Pi(b^n)$ imply $\hat{A}x \geqq \hat{b}$ and $c \cdot x \leqq \Pi(b)$. Hence $x \in \zeta(b)$.

Next it will be shown that ζ is u.h.c. if it is compact-valued for some b. It is well known in the theory of convex inequalities that if $\zeta(b_0)$ is compact for some $b_0 \in B$ then it is compact-valued for every $b \in B$. Let $g(b) = \max\{g_i(b) | i = 1, \cdots, l\}$, where $g_i(b) = \max\{x_i | \hat{A}x \geqq \hat{b},\ c \cdot x \leqq \Pi(b)\}$. The functions g_i are continuous by Theorem 1. Hence g is continuous. For each $b \in B$ define the cube

$$\psi(b) = \{x \in R^l | 0 \leqq x_i \leqq g(b), i = 1, \cdots, l\}.$$

Clearly, ψ is u.h.c. and for every x which satisfies $\hat{A}x \geqq \hat{b}$ and $c \cdot x \leqq \Pi(b)$, $\max\{x_1, \cdots, x_l\} \leqq g(b)$. Therefore $\zeta(b) \cap \psi(b) = \zeta(b)$. Hence, ζ is upper hemicontinuous according to Proposition 1.

Let $(b^n)_{n=1,\cdots}$ be a sequence in B converging to $b \in B$ and let $x \in \zeta(b)$. Then, by Proposition 2, for ζ to be lower hemicontinuous, it has to be shown that there exists a sequence $(x^n)_{n=1,\cdots}$ converging to x such that $x^n \in \zeta(b^n)$ for all n. Let x_i, $i = 1, \cdots, l$, denote the ith component of a vector $x \in R^l$ and consider the following alternative linear program.

$$\text{Minimize} \quad x_1$$

subject to

$$Ax \geqq b, \quad x \geqq 0, \quad c \cdot x \leqq \Pi(b).$$

The feasible set of this program is $\zeta(b)$ which is nonempty, closed and bounded below. Hence the program has an optimal solution. Let

$$f_1(b) = \min \{x_1 | x \geqq 0, Ax \geqq b, c \cdot x \leqq \Pi(b)\}.$$

According to Theorem 1, $f_1 : B \to R$ is a continuous function. Proceeding in the same manner define for $i = 2, \cdots, l$:

$$f_i(b) = \min \{x_i | x \geqq 0, Ax \geqq b, c \cdot x \leqq \Pi(b), e_{i-k} \cdot x \leqq f_{i-k}(b), k = 1, \cdots, i-1\},$$

where $e_i \in R^l$, $i = 1, \cdots, l$, is the usual basis for R^l.

For each $i = 2, \cdots, l$, f_i is a continuous function and the sequence $(y^n)_{n=1,\cdots}$ defined by $y^n = (f_1(b^n), \cdots, f_l(b^n))$ converges to $y = (f_1(b), \cdots, f_l(b))$ with $y^n \in \zeta(b^n)$ for all n and $c \cdot y = \Pi(b)$. Consider the sequence $(z^n)_{n=1,\cdots}$ defined by $z^n = x - y + y^n$. Clearly $z^n \to x$ and for all n, $c \cdot z^n = c \cdot y^n = \Pi(b^n)$. For each $j \in \{1, \cdots, m + l\}$ it follows that for all n one and only one of the following inequalities will hold:

(1) $\hat{a}_j \cdot z^n = \hat{a}_j \cdot y^n$ if and only if $\hat{a}_j \cdot x = \hat{a}_j \cdot y$,

(2) $\hat{a}_j \cdot z^n > \hat{a}_j \cdot y^n$ if and only if $\hat{a}_j \cdot x > \hat{a}_j \cdot y$,

(3) $\hat{a}_j \cdot z^n < \hat{a}_j \cdot y^n$ if and only if $\hat{a}_j \cdot x < \hat{a}_j \cdot y$.

Let J_1, J_2, J_3 denote the three subsets of $\{1, \cdots, m + l\}$ defined by (1)–(3) respectively. Consider a sequence $(\lambda^n)_{n=1,\dots}$, $0 \leqq \lambda^n \leqq 1$, converging to one and a sequence of vectors $(x^n)_{n=1,\cdots}$ defined by $x^n = \lambda^n z^n + (1 - \lambda^n) y^n$. Clearly, for any sequence $(\lambda^n)_{n=1,\cdots}$ with the above properties, $x^n \to x$, $c \cdot x^n = c \cdot y^n = \Pi(b^n)$ and for $j \in J_1 \cup J_2$, $\hat{a}_j \cdot x^n \geqq \hat{a}_j \cdot y^n \geqq \hat{b}_j^n$, where

$$\hat{b}^n = \begin{pmatrix} b^n \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}.$$

Since $y^n \to y$, $x \in \zeta(b)$, and since for all $j \in J_3$, $\hat{a}_j \cdot x < \hat{a}_j \cdot y$, there exists an n_0 such that for all $n \geqq n_0$, $\hat{a}_j \cdot y^n > \hat{b}_j^n$, $j \in J_3$.

Consider for $j \in J_3$ and $n \geqq n_0$,

$$\begin{aligned} \hat{a}_j \cdot x^n - \hat{b}_j^n &= \hat{a}_j \cdot (\lambda^n z^n + (1 - \lambda^n) y^n) - \hat{b}_j^n \\ &= \lambda^n \hat{a}_j \cdot x - \lambda^n \hat{a}_j \cdot y + \hat{a}_j \cdot y^n - \hat{b}_j^n. \end{aligned}$$

Hence $\hat{a}_j \cdot x^n - \hat{b}_j^n \geqq 0$ if and only if

$$\frac{\hat{a}_j \cdot y^n - \hat{b}_j^n}{\hat{a}_j \cdot y - \hat{a}_j \cdot x} \geqq \lambda^n, \qquad n \geqq n_0, \quad j \in J_3.$$

The left-hand side is strictly positive for $n \geqq n_0$ and converges to a value greater than or equal to one since $y_n \to y$, $\hat{b}^n \to \hat{b}$ and $\hat{a}_j \cdot x \geqq \hat{b}_j$, $j \in J_3$.

Hence define $(\lambda^n)_{n=1,\cdots}$ by

$$\lambda^n = \begin{cases} 0 & \text{for } n < n_0, \\ \min\left\{ \min\left\{ \dfrac{\hat{a}_j \cdot y^n - \hat{b}_j^n}{\hat{a}_j \cdot y - \hat{a}_j \cdot x} \,\middle|\, j \in J_3 \right\}, 1 \right\} & \text{for } n \geqq n_0. \end{cases}$$

Then for all n, $0 \leqq \lambda^n \leqq 1$, $\lambda^n \to 1$. Moreover, $x^n \to x$ and for all $j \in \{1, \cdots, m + l\}$ and all n, $\hat{a}_j \cdot x^n \geqq \hat{b}_j^n$, i.e., $x^n \geqq 0$, $Ax^n \geqq b^n$ and $c \cdot x^n = \Pi(b^n)$. Hence $x^n \in \zeta(b^n)$. Q.E.D.

The significance of the strong continuity property of the correspondence ζ stems from the fact that every closed and convex-valued lower hemicontinuous correspondence admits a continuous selection (see, e.g., Michaels [7]), i.e., there exists a continuous function $f: B \to R^l$ such that for all $b \in B$, $f(b) \in \zeta(b)$. In fact, the function f constructed in the proof has this property.

Remark. An immediate application of Theorem 2 yields a new continuity result in game theory. It is well known that the core of a side payment game in characteristic function form for a finite set of players is the optimal policy set of a certain linear program. Hence, the core viewed as a set-valued mapping from the set of characteristic functions for which the core exists to the set of payoffs is a continuous correspondence.

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