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THE LIMIT OF THE CORE OF AN ECONOMY WITH PRODUCTION*

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1. INTRODUCTION

THE BEHAVIOR OF THE CORE of large economies has received considerable attention during the past decade during which two distinct lines of research have been followed. The first originates with Debreu and Scarf [7] who supplied a rigorous generalization and proof for a conjecture made already by Edgeworth [9], which since has entered economic textbooks as the statement that the core "shrinks" to the set of competitive equilibria if the number of economic agents becomes infinitely large. The other line of research was started by Aumann [2] who used, as a representation of a large economy, an atomless measure space. Both approaches represent an attempt to describe the intuitive phenomenon of competition, that the power of any individual agent to influence the outcome of trading diminishes if there are sufficiently many other agents who also participate in the market. Recently, Hildenbrand [12] indicated that an atomless economy may be considered as the limit of a sequence of certain increasing but finite economies which provides a link between the two approaches.

Most of the available results, however, deal only with pure exchange economies, except for Arrow and Hahn [1], Debreu and Scarf [7], Champsaur [6], and Hildenbrand [10] where production is treated in a very special way. Recently, the general case of the core with production has been formalized and existence proofs were given for the finite case in [3] and for the approach in a measure space by Sondermann in [14]. In [4] and [14] similar market equilibrium concepts were proposed which allow a comparison of the set of equilibrium allocations with the core. Since then a conjecture has been formulated which asserts that the same results may be obtained as in the case of pure exchange. This paper intends to show that this is, in principal, the case for the result of Debreu and Scarf with very general assumptions about the technology distribution. For the case of an atomless economy, however, the example in [5] indicates that the identity of the core and the set of equilibrium allocations may not be expected if the production correspondence is not strictly additive.

2. A LIMIT THEOREM

Let $I_1 = \{1, \ldots, m\}$ be a finite set of consumers (the set of types) with the associated characteristics $(X_i, e_i, \succsim_i)$ $i = 1, \ldots, m$. Let I_r, $r \geqq 2$, denote the

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r-fold replica of I_1 and consider the economy

$$E_r = \{I_r, (X_i), (e_i), (\underset{i}{\succsim}), ((Y(S)), Y_r)\}$$

where $((Y(S)), Y_r)$, $S \subset I_r$, describes the technology distribution of the economy E_r. E_r consists of $m \cdot r$ consumers which will be indexed by the pair (i, q), $i = 1, \ldots, m$ and $q = 1, \ldots, r$. An allocation of E_r is an $m \cdot r$ — list of vectors $x_{iq} \in R^l$ such that $x_{iq} \in X_i$, $i = 1, \ldots, m$ for all $q = 1, \ldots, r$. An allocation of E_r is feasible if there exists a $y \in Y_r$ such that

$$\sum_{i=1}^{m} \sum_{q=1}^{r} x_{iq} = r \sum_{i=1}^{m} e_i + y .$$

For any economy E_r an allocation (x_{iq}) is said to be blocked by a coalition $S \subset I_r$ if there exist x'_{iq}, $(i, q) \in S$ and $y' \in Y(S)$ such that

(1) $$x'_{iq} \underset{i}{\succsim} x_{iq} \qquad \text{for all } (i, q) \in S$$

$$x'_{iq} \underset{i}{\succ} x_{iq} \qquad \text{for some } (i, q) \in S$$

(2) $$\sum_{(i,q) \in S} x_{iq} = \sum_{(i,q) \in S} e_i + y' .$$

Then the core of the economy E_r, $\mathscr{C}(E_r)$, is the set of feasible allocations of E_r which no coalition $S \subset I_r$ can block.

The following assumptions will be needed to prove the main result of this section. For all $i \in I_1$

ASSUMPTION (C1). $X_i = R^l_+$, $e_i \gg 0$.

ASSUMPTION (C2). local non-satiation holds.

ASSUMPTION (C3). $\succsim_i$ is continuous and strongly convex, i.e., for all x_i and x'_i such that $x_i \succsim_i x'_i$ and all α, $0 < \alpha < 1$, $\alpha x_i + (1 - \alpha) x'_i \succ_i x'_i$.

Let Y_1, the technology of the basic economy, be a closed convex subset of R^l containing the origin. For the production possibilities of coalitions it is assumed that for any $r = 1, \ldots$

ASSUMPTION (P1). $S \subset I_r$ implies $Y(S) \subset Y_r$.

ASSUMPTION (P2). $Y_r = rY_1$.

ASSUMPTION (P3). For any coalition S with $k_i(S)$ members of type i and $k_i(S) \geqq 1$ for all $i \in I_1$, $\operatorname{Min} \{k_i(S) \mid i \in I_1\} Y_1 \subset Y(S)$.

Assumptions (C1)–(C3) are the same as in [7] and [8]. (P1) indicates that no coalition within a given economy should be able to produce something which the economy as a whole cannot produce. (P2) implies strict additivity of the production correspondence for coalitions with equal numbers of all types of consumers. Such an assumption seems natural if the replica process is viewed as purely increasing the number of traders and trading possibilities by replicating types of economies with no new effect on production possibilities in the aggregate. This excludes a phenomenon which may best be described as increasing returns to size. In view of the example in [**5**] it is immediate that the exclusion of increasing returns to size is essential to obtain an equivalence result for the core and the set of equilibria in the limit.

(P3) is a monotonicity assumption, stating that for all coalitions with at least one member of each type the addition of new members does not decrease production possibilities, i.e., any such coalition can produce at least as much as the largest proper subeconomy it contains. It also implies that production possibilities actually increase if at least one consumer of each type is added to a coalition. It should be noted, however, that (P1)–(P3) do not imply additivity over partitions for any finite as well as for the limit economy. Finally, if Y_1 is a convex cone then (P1)–(P3) imply that for all S, $Y(S) = Y_1$ which is the assumption made by Debreu and Scarf.

LEMMA 1. *An allocation in the core of E_r, $r \geqq 2$, assigns the same commodity bundle to all consumers of the same type.*

This is the standard equal treatment result and the proof is identical to the one in [**7**]. Lemma 1 implies that an allocation in the core for the sequence of economies E_r can be described by an m-list of commodity vectors $(x_1, \ldots, x_m)$, as in [**7**] and [**8**], which in turn yields that the representation C_r of the core as a set in $R^{l \cdot m}$ is a decreasing sequence in $R^{l \cdot m}$ for $r = 1, \ldots$.

For the desired comparison of the core with the appropriate equilibrium the following definition of an equilibrium with a stable profit distribution (see [**4**] and [**14**]) is needed. Let $p \in R^l$, $p \neq 0$ be a price system and let (t_{iq}), $(i, q) \in I_r$, be a list of real numbers describing profit payments which consumer (i, q) receives.

DEFINITION. A list $((x_{iq}), (t_{iq}), p, y)$ is an equilibrium with a stable profit distribution for the economy E_r if

(1) $$\sum_{(i,q) \in I_r} x_{iq} = r \sum_{i \in I_1} e_i + y, \qquad y \in Y_r$$

i.e., (x_{iq}) is a feasible allocation

(2) $$t_{iq} \geqq 0, \qquad \text{for all } (i, q) \in I_r$$

(3) x_{iq} is a best element with respect to $\succsim_i$ in consumer (i, q)'s budget set $\{x_{iq} \in X_i \mid p \cdot x_{iq} \leqq p \cdot e_i + t_{iq}\}$

(4) $$\sum_{(i,q) \in I_r} t_{iq} = p \cdot y$$

(5) $$\sum_{(i,q) \in S} t_{iq} \geqq \sup p \cdot Y(S) \qquad \text{for all } S \subset I_r.$$

The first result is that any equilibrium with a stable profit distribution yields an allocation in the core, the proof of which is omitted here (see [**4**] and [**14**]). The main result of this paper is the following theorem.

THEOREM. *If assumptions* (C1)–(C3), (P1)–(P3) *hold, then for every*

$$(x_i) \in \bigcap_{r=1}^{\infty} C_r$$

there exist prices $p \in R^l$, $p \neq 0$, *and profit payments* $(t_i), i \in I_1$ *such that* $((x_i), (t_i), p, y)$ *is an equilibrium with a stable profit distribution for all* E_r.

For every $i \in I_1$, and any $x_i \in X_i$ define $\Gamma_i(x_i) = \{z_i | z_i + e_i \succ_i x_i\}$. Let $k_p = (k_{1p}, \ldots, k_{mp})$ denote a list of strictly positive integers representing the profile p of a coalition with k_{ip} members of type i. For an allocation $x = (x_1, \ldots, x_m)$ in the core of E_r define for any k_p such that $k_{ip} \leqq r, i \in I_1$,

$$\Gamma(k_p, x) = \sum_{i \in I_1} k_{ip} \Gamma_i(x_i)$$

which is a non-empty and convex set in R^l. Finally, let x denote an allocation in the core for every $r \geqq 1$ and define $\Gamma(x) = \operatorname{conv} \bigcup_{k_p} \Gamma(k_p, x)$. According to a theorem by Rockafellar [**13**, (18, Theorem 3.3)] $\Gamma(x)$ may be written as

$$\Gamma(x) = \bigcup \{ \sum_{j \in J} \lambda_j \Gamma(k_j, x) \}$$

where the union is taken over all finite proper convex combinations.

LEMMA 2. *Let* $x \in \bigcap_{r=1}^{\infty} C_r$. *Then* $Y_1 \cap \Gamma(x) = \emptyset$.

PROOF. Suppose the assertion were false. Then there exist a $y \in Y_1$, a finite set $J = \{1, \ldots, n\}$, associated numbers $\lambda_j > 0$, profiles k_j, and commodity allign bundles $z_{ij} \in \Gamma_i(x_i)$ such that $\sum_{j \in J} \lambda_j = 1$ and

$$y = \sum_{j \in J} \lambda_j \sum_{i \in I_1} k_{ij} z_{ij} .$$

Let q be an integer and define a commodity bundle z_i^q, $i = 1, \ldots, m$

$$z_i^q = \frac{q}{a_i^q} \sum_{j \in J} \lambda_j k_{ij} z_{ij}$$

where a_i^q is the smallest integer greater than or equal to $q \sum_{j \in J} \lambda_j k_{ij}$. Clearly, for $q \to \infty$ z_i^q converges to the point

$$z_i = \frac{\sum_{j \in J} \lambda_j k_{ij} z_{ij}}{\sum_{j \in J} \lambda_j k_{ij}},$$

a convex combination of the points $z_{ij} \in \Gamma_i(x_i)$, $j = 1, \ldots, n$. Hence $z_i \in \Gamma_i(x_i)$ and for q sufficiently large $z_i^q + e_i \succ_i x_i$, $i \in I_1$. Now consider a coalition of a_i^q members of type i, $i = 1, \ldots, m$, and assign to each member of type i the allocation $z_i^q + e_i$. Then

$$\begin{aligned} \sum_{i \in I_1} a_i^q (z_i^q + e_i) &= \sum_{i \in I_1} q \sum_{j \in J} \lambda_j k_{ij} z_{ij} + \sum_{i \in I_1} a_i^q e_i \\ &= q y + \sum_{i \in I_1} a_i^q e_i . \end{aligned}$$

Since $\sum_{j\in J}\lambda_j = 1$, $\sum_{j\in J}\lambda_j k_{ij} \geqq 1$ for all $i \in I_1$. Hence the choice of a_i^q, namely $q\sum_{j\in J}\lambda_j k_{ij} \leqq a_i^q$, implies $q \leqq \mathrm{Min}_{i\in I_1} a_i^q$. Since $y \in Y_1$, $qy \in Y(S)$ due to assumption (P3). Hence for large q the coalition S with profile (a_i^q) will block the allocation x using the allocation $(a_i^q(z_i^q + e_i))$ which is feasible for S, contradicting that $x \in \bigcap_{r=1}^{\infty} C_r$. Therefore, $Y_1 \cap \Gamma(x) = \emptyset$.

Since Y_1 and $\Gamma(x)$ are convex there exists a price vector $p \in R^l$, $p \neq 0$ such that for all $y \in Y_1$, $p\cdot y \leqq 0$ and for all $z \in \Gamma(x)$, $p\cdot z \geqq 0$. $p\cdot Y_1 \leqq 0$ implies $p\cdot(r\ Y_1) \leqq 0$ for all $r = 1, \ldots$ which in turn yields $\sup p\cdot Y(S) \leqq \sup p\cdot(r\ Y_1) \leqq 0$ for all $r = 1, \ldots$ and all $S \subset I_r$.

Next it will be shown that for all $i \in I_1$, $p\cdot\Gamma_i(x_i) \geqq 0$. Suppose there exist $z_{i'} \in \Gamma_{i'}(x_{i'})$ for some fixed i' such that $p\cdot z_{i'} < 0$. Choose fixed integers k_i, $z_i \in \Gamma_i(x_i)$, $i \neq i'$. For $k_{i'}$ large enough

$$\sum_{i\neq i'} p\cdot(k_i z_i) + k_{i'} p\cdot z_{i'} < 0$$

contradicting that $p\cdot\Gamma(k, x) \geqq 0$ for all k.

Now choose as profit payments $t_i = 0$ for all $i \in I_1$. Then $p\cdot z_i \geqq 0$ for all $z_i \in \Gamma_i(x_i)$, $i \in I_1$, implies that for any $x_i' = z_i + e_i \succ_i x_i$, $p\cdot x_i' \geqq p\cdot e_i + t_i$. Since $i \in I_1$ is locally not satiated one also has $p\cdot x_i \geqq p\cdot e_i$. Furthermore,

$$\sum_{i\in I_1} x_i = \sum_{i\in I_1} e_i + y$$

with $p\cdot y \leqq 0$ yields $p\cdot x_i = p\cdot e_i$ and $p\cdot y = 0$. Standard arguments then show that for any $x_i' \succ_i x_i$, $p\cdot x_i' > p\cdot e_i + t_i$ since $e_i \gg 0$. Hence $((x_i)(t_i), p, y)$ is an equilibrium with a stable profit distribution which completes the proof of the theorem.

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