Chapter 7

EXISTENCE OF EQUILIBRIA WITH PRICE REGULATION*

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1. Introduction

Economies with increasing returns industries in general fail to have competitive equilibria and, therefore, the general issue to discuss alternative equilibrium concepts and their optimality properties has been considered by economists for a long time. Since the prices for an increasing returns industry, or more specifically for a public monopoly, cannot be competitive much of the discussion has centered around characterizing first-best or second-best Pareto optimal allocations by linear prices [Guesnerie (1975)] or by non-linear prices [Brown and Heal (1979)].

Apart from the discussion of optimal pricing schemes, the question of existence of an equilibrium has received additional attention in the past. The paper by Dierker, Guesnerie and Neuefeind (1983) contains an extensive discussion of some of the crucial problems involved in proving existence. Many of these problems stem from the application of the marginal cost pricing principle in one form or another. In particular, they arise when the monopoly's production level becomes small. The wide acceptance of the marginal cost pricing principle in applications as well as in theoretical discussions seems to be based on an implicit supposition that it implies some form of second-best Pareto optimality. It still remains an open question today to what extent these two characteristics are linked with each other. For an existence result with sufficiently wide applicability, prices have to deviate from marginal cost in many cases at low levels of production. Therefore, the pricing problem will be formulated in a way which avoids these boundary problems and without using an illegitimate justification for optimality. Moreover, the differentiability prop-

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erties required to apply the principle seem to be unnecessarily strong for existence purposes. The opposite approach is taken by Brown and Heal (1982), Beato (1982) and Cornet (1984).

In models of market economies, the information available to a public enterprise consists typically of its own technology, the prices of its input factors and of the demand for its outputs. It is natural, therefore, to characterize the price setting behavior in a way which depends on these parameters. The recent literature on cost sharing rules describes pricing functions of this type, i.e., where the available information consists of an aggregate demand vector and the cost structure (or technology) of the enterprise [see e.g. Mirman and Tauman (1982), Bös and Tillmann (1982), and others]. Some of the papers of this literature also treat the general existence problem for a particular pricing rule and for a special simple version of a general equilibrium model of an economy. The research by Dierker, Guesnerie and Neuefeind (1983), which was carried out independently, reports an existence result similar in spirit but technically different from the one presented here. Moreover, some of their assumptions for the public sector's technology are stronger than the ones used here.

It is the object of this note to characterize a general class of pricing rules which uses aggregate data only and to provide an existence proof for an equilibrium of an economy using such a rule for the public sector, but which is otherwise competitive in the usual sense. This generalizes the existing literature in several ways. Factor prices for the public enterprise are determined endogenously for arbitrary (finite) numbers of public outputs and inputs. The class of pricing rules is sufficiently general to cover a wide range of cases. Among them are different modified versions of the marginal cost pricing rule, the so called axiomatic rules like the Aumann–Shapley prices, and other game theoretic solutions, as well as some variants of mark-up and full cost pricing.

2. *The model*

Consider an economy consisting of a set $I = \{1, \dots, i, \dots, m\}$ of consumers and a set $J = \{0, 1, \dots, j, \dots, n\}$ of producers where $j = 1, \dots, n$ denote private competitive firms and $j = 0$ denotes a public enterprise. The list of commodities can be partitioned into a list of l commodities produced by the public enterprise and a list of k commodities required as inputs by the public enterprise. Let $H = \{1, \dots, l, l+1, \dots, l+k\}$ denote the set of commodities and $\mathbf{R}^H = \mathbf{R}^l \times \mathbf{R}^k$ the commodity space. A price system for the economy is a pair $(p, w) \in \mathbf{R}^H$ with $p \in \mathbf{R}^l_+$ and $w \in \mathbf{R}^k_{+}$.

For any producer $j \in J$, let $Y_j \subset \mathbf{R}^H$ denote its production possibility set with the usual sign convention for inputs (non-positive) and outputs (non-negative). For $j = 0$ it follows that $\mathrm{proj}_l Y_0 \subset \mathbf{R}^l_+$ and $\mathrm{proj}_k Y_0 \subset -\mathbf{R}^k_+$, where proj_l (resp. proj_k) are the projections on the first l (resp. last k) coordinates of $\mathbf{R}^H$. Each competitive firm $j = 1,\ldots, n$ maximizes profits subject to given prices, i.e.,

$$\pi_j(p, w) = \max\{(p, w)\cdot y_j | y_j \in Y_j\},$$

and

$$\eta_j(p, w) = \{y_j \in Y_j | (p, w)\cdot y_j = \pi_j(p, w)\}.$$

η_j is the net supply correspondence of producer j ($j = 1,\ldots, n$). It should be noted that there is no restriction on competitive production as to which commodities are inputs or outputs. Thus, in particular, a competitive firm j may produce some or all of the commodities which the public enterprise produces.

For the public enterprise $j = 0$, consider the set of possible output bundles $A = \mathrm{proj}_l Y_0$. For any $\alpha \in A$, define the input requirement set to produce α as $\tilde{Y}_0(\alpha) = \{y_k \in \mathbf{R}^k | (\alpha, y_k) \in Y_0\}$. Then, cost minimization yields the cost function $C(\alpha, w)$ given by

$$C(\alpha, w) = \min\{-wy_k | y_k \in \tilde{Y}_0(\alpha)\},$$

and the factor demand correspondence

$$\eta_0(\alpha, w) = \{y_k \in \tilde{Y}_0(\alpha) | -wy_k = C(\alpha, w)\}.$$

Given some weak regularity properties on Y_0 and $\tilde{Y}_0$, C and η_0 are well defined on $A \times \mathbf{R}^k_+$. Then, it is known from standard production theory [e.g., Varian (1984, p. 44)] that

(i) $C(.,.)$ is non-decreasing in (α, w), continuous if $w \gg 0$, positively linear homogeneous and concave in w.
(ii) $\eta_0(.,.)$ is upper hemi-continuous in (α, w), homogeneous of degree zero in w, and $\eta_0(.)$ is convex valued if $\tilde{Y}_0(.)$ is.

In the model under consideration, prices $p \in \mathbf{R}^l_+$ for the publicly produced commodity are determined by the public enterprise whereas all other prices $w \in \mathbf{R}^k_+$ are assumed to arise out of a competitive market mechanism. Therefore, they are taken as given by the public enterprise. A pricing decision p in a given situation depends in general on the objective and on the technology of the public enterprise. As is well known from the second-best literature, optimal pricing rules require information about individuals' preferences. Moreover, in some cases these may imply negative profits of the public enterprise which in turn requires some form of a lump sum taxation scheme on private agents. In the context of this paper, the pricing decision will be characterized under the assumption that costs and aggregate demand for the outputs of the public enterprise are the only information available. As an objective or as a constraint one may impose that the public enterprise obtains a certain minimal profit level. In fact the cost sharing literature imposes a zero profit condition. However, in a non-monetary economy with relative prices only, a positive nominal profit level has little economic significance. Therefore, a relative profit measure $\lambda \geqq 0$ will be used here which is the relative excess of revenue over cost for the public enterprise.

Formally, a pricing rule can be defined as follows:

Definition 1. For a given cost function $C: \mathbf{R}^l_+ \times \mathbf{R}^k_+ \rightarrow \mathbf{R}$ and a required profit level $\lambda \geqq 0$, a correspondence

$$F: \mathbf{R}^l_+ \times \mathbf{R}^k_+ \rightarrow \mathbf{R}^l$$

is a pricing rule for C and λ if for all $(\alpha, w) \in \mathbf{R}^l_+ \times \mathbf{R}^k_+$

$$p \in F(\alpha, w) \quad \text{implies} \quad p \cdot \alpha = (1 + \lambda)C(\alpha, w).$$

In the most general context F may vary also with C and λ. However, here C and λ are taken as given. Hence aggregate demand α and input prices w are the only relevant determining variables.

To complete the description of the economy, it will be assumed that each consumer $i = 1, \ldots, m$ is characterized by a list $(X_i, \succsim_i, e_i, (\theta_{ij}))$, where $X_i \subset \mathbf{R}^H$ is the consumption set, $\succsim_i$ are the preferences, $e_i \in \mathbf{R}^H_+$ are i's endowments. $\theta_{ij} \geqq 0$ is i's profit share of producer $j = 0, \ldots, n$ with the usual condition that $\sum_{i \in I} \theta_{ij} = 1$ for all $j \in J$. Each consumer i maximizes his preferences $\succsim_i$ given his budget constraint which yields consumer i's excess demand correspondences as

$$\xi_i(p, w, \pi_0) = \{ z_i \in \beta_i(p, w, \pi_0) | e_i + z_i' \precsim_i e_i + z_i \text{ for all } z_i' \in \beta_i(p, w, \pi_0) \},$$

where

$$\beta_i(p, w, \pi_0) = \left\{ z_i \in \mathbf{R}^H | (p, w) z_i \leq \theta_{i0}\pi_0 + \sum_{j=1}^{n} \theta_{ij}\pi_j(p, w) \right.$$

$$\left. \text{and } e_i + z_i \in X_i \right\}.$$

Given the characteristics of economy it is now straightforward to define attainable states and the equilibrium concept.

Definition 2. A list $((z_i), (y_j), (\alpha, y_0))$ is an attainable allocation if

(i) $e_i + z_i \in X_i$ for all $i \in I$,
(ii) $y_j \in Y_j$ for all $j \in J \setminus \{0\}$,
(iii) $(\alpha, y_0) \in Y_0$,
(iv) $\sum_{i=1}^{m} z_i - (\alpha, y_0) - \sum_{j=1}^{n} y_j = 0$.

Definition 3. An attainable allocation $((z_i^*), (y_j^*), (\alpha^*, y_0^*))$ and a pair $(p^*, w^*) \in \mathbf{R}_+^l \times \mathbf{R}_+^k$, $(p^*, w^*) \neq 0$, is called an equilibrium for the pricing rule F if

(i) $p^* \in F(\alpha^*, w^*)$,
$p^* \cdot \alpha^* + w^* \cdot y_0^* = \pi_0 = \lambda C(\alpha^*, w^*)$,
(ii) $x_i^* \in \xi_i(p^*, w^*, \pi_0)$ for all $i \in I$,
(iii) $y_j^* \in \eta_j(p^*, w^*)$ for all $j \in J \setminus \{0\}$,
$y_0^* \in \eta_0(\alpha^*, w^*)$.

It should be noted that the definition of the equilibrium allows for a large set of institutional arrangements. $j = 0$ may be a natural monopoly which is privately owned as well as a publicly owned firm which distributes its profits to consumers. Since no restriction has been introduced so far on the dimensionality of the private production sets and on endowments, the publicly produced commodities could be either inputs or outputs of the competitive sector, or even held as endowments by consumers. A straightforward generalization of the model could include the regulation of certain factor prices and/or the prices of certain endowments as well.

Some care must be taken, however, when such generalizations are considered if existence of an equilibrium is to be guaranteed. In order to see this one may imagine the extreme case where regulation is introduced on factor prices but public production is identically equal to zero. Then, the only prices which will be consistent with competitive behavior and market clearing are the Walrasian prices which need not coincide with the regulated ones. Therefore, no equilibrium with price regulation exists. On the other hand any regulated prices would imply some form of rationing in order to guarantee equality of supply and demand.

The argument above has been derived for the case of factor (input) price regulation by the public monopoly. However, the same argument can be made for the case of output price regulation considered here. The key issue involved from an existence point of view is that the public enterprise should, apart from the price regulation, have sufficient influence through its own input–output decision on aggregate excess demand. It remains an open problem how to formulate minimal requirements and what the maximal number of markets controlled by the government could be to guarantee existence.

The approach chosen in the context of the present paper is to restrict price regulation to those situations where the price of a particular commodity is controlled only as long as the public enterprise is active in that market. This idea leads to a weaker notion of an equilibrium called a quasi-equilibrium relative to a given pricing rule.

Definition 4. An attainable allocation $((z_i^*), (y_j^*), (\alpha^*, y_0^*))$ and a triple $(p^*, q^*, w^*) \in \mathbf{R}_+^l \times \mathbf{R}_+^l \times \mathbf{R}_+^k$, $(p^*, w^*) \neq 0$, is called a quasi-equilibrium for the pricing rule F if

(i) $q^* \in F(\alpha^*, y_0^*)$ and $\pi_0 = \lambda C(\alpha^*, w^*)$,
(ii) $x_i^* \in \xi_i(p^*, w^*, \pi_0)$ for all $i \in I$,
(iii) $y_j^* \in \eta_j(p^*, w^*)$ for all $j \in J \setminus \{0\}$,
$y_0 \in \eta_0(\alpha^*, w^*)$,
(iv) $\alpha_h^* > 0$ implies $p_h^* = q_h^*$ for all $h = 1, \ldots, l$.

The concept of a quasi-equilibrium for a pricing rule F seems to be the natural extension of the notion of a competitive equilibrium. First, it contains the competitive equilibrium without public activity as a special case. Second, for the case of positive public production, the uniform price set by the public enterprise becomes a competitive price for all other agents, i.e., also for those producers who compete in output with the public enterprise. Hence, one would expect that in such cases, a large part of possible production inefficiencies are eliminated in equilibrium. Third, if the publicly produced commodities are desirable but neither producible by the competitive sector nor available as endowments, then any quasi-equilibrium will be an equilibrium as defined in Definition 3.

3. *Existence of equilibria*

In order to prove existence of an equilibrium with price regulation it should be clear that a general set of sufficient conditions would require conditions for the competitive sector of the economy which are similar to those needed to prove existence of a competitive equilibrium. However, nothing can be gained by making assumptions which in fact guarantee the existence of a competitive equilibrium. In this case the concept of quasi-equilibrium is too weak since every competitive equilibrium is a quasi-equilibrium for any pricing rule with an inactive public sector, provided there are no public fixed costs. Therefore, the assumptions made will be weaker than the ones sufficient to prove existence of competitive equilibria. This will avoid the case of a trivial quasi-equilibrium. On the other hand, the consistency of the competitive mechanism and of the pricing rule creates the actual problem for an existence proof in the typical situation where the public enterprise produces some desired output otherwise not available in the economy.

Define

$$\zeta(p, w, \pi_0) = \sum_{i \in I} \xi_i(p, w, \pi_0) - \sum_{j \in J \setminus \{0\}} \eta_j(p, w)$$

to be the excess demand correspondence of the competitive sector for the given profit level π_0. Then, Walras Law holds for any (p, w), i.e.,

$$z \in \zeta(p, w, \pi_0) \quad \text{implies} \quad pz^l + wz^k \leq \pi_0.$$

Let

$$W = \left\{ w \in \mathbf{R}^k_+ \mid \sum_{h=1}^{k} w_h = 1 \right\} \quad \text{and} \quad P = \mathbf{R}^l_+.$$

Theorem. Assume that the set of attainable allocations of the economy is compact and that the following conditions are satisfied:

A.1. *For each consumer $i \in I$,*

(i) $X_i \neq \varnothing$, *convex, closed, and bounded below,* $(x^l, x^k) \in X_i$ *and* $y^l \geqq x^l$ *implies* $(y^l, x^k) \in X_i$,

(ii) $e_i \in \mathbf{R}^H_+$ *and there exists* $(0, x^k_i) \in X_i$ *such that* $x^k_i \ll e^k_i$,

(iii) $\succsim_i$ *continuous, convex, and strictly monotonic.*

A.2. *For each competitive firm* $j \in J \setminus \{0\}$, $Y_j \neq \varnothing$, *convex, closed and* $0 \in Y_j$.

A.3. *For* $j = 0$,

(i) $Y_0 \neq \varnothing$, *closed, and* $A = \text{proj}_l Y_0 = R^l_+$,

(ii) $(\alpha, y_0) \in Y_0$ *and* $y \in \mathbf{R}^k_+$ *implies* $(\alpha, y_0 - y) \in Y_0$,

(iii) $\tilde{Y}_0(\alpha)$ *convex.*

A.4. *C: $A \times W \to \mathbf{R}$ is continuous and $C(0, w) = 0$, $C(\alpha, w) > 0$ for all $\alpha \gg 0$.*

A.5. *η_0: $A \times W \to -\mathbf{R}^k_+$ is convex valued and has a closed graph.*

A.6. *F: $A \times W \to \mathbf{R}^l_+$ is convex valued and has a closed graph.*

A.7. *For every sequence (α_r, w_r, q_r) such that $q_r \in F(\alpha_r, w_r)$, $(\alpha_r, w_r) \to (\bar{\alpha}, \bar{w})$ and $\bar{\alpha} \neq 0$, $\bar{q} \in F(\bar{\alpha}, \bar{w})$ implies $\bar{q}\bar{\alpha} = \lim_{h \in H^+} \sum q_r^h \alpha_r^h$, where $H^+ = \{h | \bar{\alpha}^h > 0\}$.*

Then, there exists a quasi-equilibrium with price regulation for the pricing rule F.

Proof. The set of attainable states is defined as

$$X = \left\{(z_1, \ldots, z_m, y_0, y_1, \ldots, y_n) | z_i + e_i \in X_i,\ y_j \in Y_j,\right.$$
$$\left. \sum_{i \in I} z_i - \sum_{j=1}^{n} y_j = y_0 \right\}.$$

Since X is compact, there exists a large compact, convex cube $\tilde{Z} \subset \mathbf{R}^H$ with $0 \in \operatorname{int} \tilde{Z}$ such that

$$\begin{aligned} &\operatorname{proj}_i X \subset \operatorname{int} \tilde{\tilde{Z}} \quad \text{for all} \quad i \in I, \quad \tilde{\tilde{Z}} = (n+1)/m\tilde{Z}, \\ &\operatorname{proj}_j X \subset \operatorname{int} \tilde{Z} \quad \text{for all} \quad j \in J. \end{aligned}$$

Define restricted excess demand correspondences for $i \in I$ and $j \in J \setminus \{0\}$ by

$$\begin{aligned} \tilde{\eta}_j(p, w) = \{ y \in Y_j \cap \tilde{Z} | (p, w) y = \pi_j(p, w) \\ = \max(p, w)(Y_j \cap \tilde{Z}) \}, \end{aligned}$$

$$\begin{aligned} \tilde{\xi}_i(p, w, \pi_0) = \{ z_i \in \tilde{\beta}_i(p, w, \pi_0) | e_i + z_i' \precsim_i e_i + z_i \\ \text{for all } z_i' \in \tilde{\beta}_i(p, w, \pi_0) \}, \end{aligned}$$

where

$$\tilde{\beta}_i(p, w, \pi_0) = \left\{ z_i \in \tilde{\tilde{Z}} | e_i + z_i \in X_i \text{ and} \right.$$
$$\left. (p, w) z_i \leqq \theta_{i0}\pi_0 + \sum_{j=1}^{n} \theta_{ij}\pi_j(p, w) \right\}.$$

Define the restricted excess demand correspondence of the competitive sector as

$$\tilde{\zeta}(p, w, \pi_0) = \sum_{i \in I} \tilde{\xi}_i(p, w, \pi_0) - \sum_{j=1}^{n} \tilde{\eta}_j(p, w).$$

Let $Z = n\tilde{Z} + m\tilde{\tilde{Z}}$. Then, it is straightforward to show that $\tilde{\zeta}$: $P \times W \times \mathbf{R}_+ \to Z$ is u.h.c. with non-empty and convex values for all (p, w, π_0).

The remainder of the proof will proceed in two steps. First, a fixed point which possesses most of the equilibrium properties will be shown to exist for appropriately chosen compact price sets P^r. The second step uses a limit argument to establish existence when P^r tends to $\mathbf{R}^l_+$.

Let $M = \sup\{\|x\| \mid x \in \operatorname{proj}_0 X\}$, $\hat{z} = M(1, \ldots, 1) \in \mathbf{R}^l_+$ and $\hat{C} = \max\{C(\hat{z}, w) | w \in W\}$. For $r = 1, \ldots,$ define as the price set $P^r = \{p \in \mathbf{R}^l_+ | p^h \leqq r(1 + \lambda)\hat{C},\ h = 1, \ldots, l\}$. The choice of M, $\hat{z}$, $\hat{C}$, and the definition of P^r implies that $F(\alpha, w) \subset P^r$ for all $r = 1, \ldots,$ $w \in W$, and $r^{-1}(1, \ldots, 1) \leqq \alpha \leqq \hat{z}$. For any $z \in Z \subset \mathbf{R}^{l+k}$, define

$$z^l_+ = \left(\max\{0, z^h\}\right)^l_{h=1} \quad \text{and} \quad z^l_* = \left(\max\{r^{-1}, z^h\}\right)^l_{h=1}.$$

Finally, let $Z_0 = \operatorname{proj}_K \tilde{Z}$ where $K = \{h \in H | h = l+1, \ldots, l+k\}$. For $r = 1, \ldots,$ consider the correspondence ψ^r: $P^r \times P^r \times W \times Z \times Z_0 \to P^r \times P^r \times W \times Z \times Z_0$ whose components are defined by

$$\psi^r_1(p, q, w, z, y_0) = \arg\max\{\tilde{p}z^l | 0 \leqq \tilde{p} \leqq q\},$$
$$\psi^r_2(p, q, w, z, y_0) = F(z^l_*, w),$$
$$\psi^r_3(p, q, w, z, y_0) = \arg\max\{v(z^k - y_0) | v \in W\},$$
$$\psi^r_4(p, q, w, z, y_0) = \tilde{\zeta}(p, w, \lambda C(z^l_*, w)),$$
$$\psi^r_5(p, q, w, z, y_0) = \eta_0(z^l_*, w).$$

It is straightforward to verify that for each $r = 1, \ldots,$ ψ^r fulfills the requirements of Kakutani's Fixed Point Theorem.

Hence, there exists $(\bar{p}, \bar{q}, \bar{w}, \bar{z}, \bar{y}_0)$ such that

(i) $\bar{p} \in \arg\max\{\tilde{p}\bar{z}^l | 0 \leqq \tilde{p} \leqq \bar{q}\}$,
(ii) $\bar{q} \in F(\bar{z}^l_*, \bar{w})$,
(iii) $\bar{w} \in \arg\max\{v(\bar{z}^k - \bar{y}_0) | v \in W\}$,
(iv) $\bar{z} \in \tilde{\zeta}(\bar{p}, \bar{w}, \lambda C(\bar{z}^l_*, \bar{w}))$,
(v) $\bar{y}_0 \in \eta_0(\bar{z}^l_*, \bar{w})$.

(ii), (iv) and (v) yield

$$\bar{q}\bar{z}_*^l + \bar{w}\bar{y}_0 = \lambda C(\bar{z}_*^l, \bar{w}) \quad \text{and} \quad \bar{p}\bar{z}^l + \bar{w}\bar{z}^k \leqq \lambda C(\bar{z}_*^l, \bar{w}),$$

and therefore, applying (i),

(vi) $\bar{w}(\bar{z}^k - \bar{y}_0) \leqq \bar{q}\bar{z}_*^l - \bar{p}\bar{z}^l = \bar{q}(\bar{z}_*^l - \bar{z}_+^l)$.

Suppose for some r, $\bar{z}^l \geqq \bar{z}_*^l$. Then $\bar{z}^l = \bar{z}_*^l \gg 0$ and $\bar{p} = \bar{q}$ because of (i). Moreover, (vi) together with (iii) implies that $\bar{z}^k - \bar{y}_0 \leqq 0$. Using the free disposal assumption of A.3 yields $\bar{z} \in Y_0$. It follows that there exists an attainable allocation $((\bar{z}_i), (\bar{y}_j), \bar{z})$ such that

(vii) $\bar{z}_i \in \tilde{\xi}_i(\bar{p}, \bar{w}, \pi_0)$ and $\bar{z}_i \in \operatorname{int} \tilde{Z}$ for all $i \in I$,
(viii) $\bar{y}_j \in \tilde{\eta}_j(\bar{p}, \bar{w})$ and $\bar{y}_j \in \operatorname{int} \tilde{Z}$ for all $j \in J \setminus \{0\}$,

with $\bar{z} = \sum_{i \in I} \bar{z}_i - \sum_{j \in J \setminus \{0\}} \bar{y}_j \in \operatorname{int} Z$.

From (vii) and A.1 one obtains $\bar{p} \gg 0$ and from the convexity of $\gtrsim_i$

(vii′) $z_i \in \xi_i(\bar{p}, \bar{w}, \pi_0)$ for all $i \in I$.

In a similar fashion, A.2 implies

(viii′) $y_j \in \eta_j(\bar{p}, \bar{w})$.

Hence, $((z_i), (y_j), \bar{z}, (\bar{p}, \bar{w}))$ is an equilibrium for F.

Assume that $z_{*r}^l \neq z_{+r}^l$ holds for the sequence of fixed points $(p_r, q_r, w_r, z_r, y_{0r})_{r=1}^{\infty}$. Since (w_r, z_r, y_{0r}) is bounded, consider a subsequence also denoted (w_r, z_r, y_{0r}) converging to $(\bar{w}, \bar{z}, \bar{y}_0)$. Define $H^+ = \{h \in \{1, \ldots, l\} | \bar{z}^h > 0\}$ and $\bar{q} \in F(\bar{z}_+^l, \bar{w})$. From (vi) one has for all r of the subsequence

$$\begin{aligned} w_r(z_r^k - y_{0r}) &\leqq q_r(z_{*r}^l - z_{+r}^l) \\ &= (1+\lambda)C(z_{*r}^l, w_r) - \bar{q}z_{+r}^l + \bar{q}z_{+r}^l - q_r z_{+r}^l \\ &\leqq (1+\lambda)C(z_{*r}^l, w_r) - \bar{q}z_{+r}^l + \bar{q}z_{+r}^l - \sum_{h \in H^+} q_r^h z_{+r}^h. \end{aligned}$$

Clearly, $\lim z_{*r}^l = \lim z_{+r}^l = \bar{z}_+^l$ and $\bar{z}_+^l = \bar{z}^l$. The right-hand side converges to zero because of A.4 and A.7. Therefore $\bar{w}(\bar{z}^k - \bar{y}_0) = 0$, which yields, as above, $\bar{z}^k - \bar{y}_0 \leqq 0$ and $\bar{z} \in Y_0$. Hence, $\bar{z} \in \operatorname{int} Z$ and there exist converging sequences $z_{ir} \to \bar{z}_i \in \operatorname{int} \tilde{Z}$ and $y_{jr} \to \bar{y}_j \in \operatorname{int} \tilde{Z}$ such that for all $r = 1, \ldots,$

$$\begin{aligned} z_{ir} &\in \tilde{\xi}_i(p_r, w_r, \lambda C(z_{*r}^l, w_r)) && \text{for all} \quad i \in I, \\ y_{jr} &\in \tilde{\eta}_j(p_r, w_r) && \text{for all} \quad j \in J \setminus \{0\}. \end{aligned}$$

As a consequence one has for r sufficiently large:

$$z_{ir} \in \xi_i\left(p_r, w_r, \lambda C\left(z^l_{*r}, w_r\right)\right) \quad \text{for all} \quad i \in I,$$
$$y_{jr} \in \eta_j(p_r, w_r) \quad \text{for all} \quad j \in J \setminus \{0\}.$$

Moreover A.1 implies $p_r \gg 0$.

To complete the proof the two cases whether (q_r) is bounded or unbounded have to be distinguished. If (q_r) is bounded, then (p_r) is bounded. Therefore, any converging subsequence and its accumulation point $(\bar{p}, \bar{q}, \bar{w}, \bar{z}, \bar{y}_0)$ with $\bar{z} = \sum_{i \in I} \bar{z}_i - \sum_{j \in J \setminus \{0\}} \bar{y}_j$ imply the conditions above. Assumption A.1 guarantees $\bar{p} \gg 0$, and $\bar{p} \in \arg\max\{p\bar{z}^l | 0 \leqq p \leqq \bar{q}\}$ implies $\bar{p}^h = \bar{q}^h$ if $\bar{z}^h > 0$. Hence, $((\bar{z}_i), (\bar{y}_j), (\bar{z}_+, y_0), (\bar{p}, \bar{q}, \bar{w}))$ is a quasi-equilibrium.

Suppose $\|q_r\| \to \infty$ and consider the sequence of prices $(\tilde{p}_r, \tilde{w}_r) = (1/\|q_r\|)(p_r, w_r)$ which converges to $(\tilde{p}, 0)$. Since the optimizing behavior of consumers and of producers is homogeneous of degree zero in prices and the cost function is homogeneous of degree one in w, one obtains for all r

$$z_{ir} \in \tilde{\xi}_i\left(\tilde{p}_r, \tilde{w}_r, \lambda C\left(z^l_{*r}, \tilde{w}_r\right)\right) \quad \text{for all} \quad i \in I,$$
$$y_{jr} \in \tilde{\eta}_j(\tilde{p}_r, \tilde{w}_r) \quad \text{for all} \quad j \in J \setminus \{0\},$$

and for r sufficiently large

$$z_{ir} \in \xi_i\left(\tilde{p}_r, \tilde{w}_r, \lambda C\left(z^l_{*r}, \tilde{w}_r\right)\right) \quad \text{for all} \quad i \in I.$$

A.1 implies that $\tilde{p} \gg 0$. It follows that $H^+ = \emptyset$, since (i) yields for all r and $h = 1, \ldots, l$

$$p^h_r z^h_r \leqq q^h_r z^h_{*r} \leqq (1 + \lambda) C(z_{*r}, w^r),$$

which in turn implies $\bar{z}^h = 0$, since

$$0 \leqq \tilde{p}^h \bar{z}^h = \lim \frac{1}{\|q_r\|} p^h_r z^h_r \leqq \lim \frac{1}{\|q_r\|}(1 + \lambda) C(z_{*r}, w^r) = 0.$$

But then $((\bar{z}_i), (\bar{y}_j), 0, (\tilde{p}, \bar{q}, 0))$ is a quasi-equilibrium for F with $\tilde{p} \gg 0$ and any $\bar{q} \in F(0,0)$. □

The last steps in the proof indicate the importance of some positive public production. If this is not the case, then the associated quasi-equilibrium is a competitive equilibrium with zero prices for all non-public commodities. However, if preferences and/or competitive technologies imply that demand becomes unbounded if one non-public price tends to zero, then this would imply strictly positive equilibrium prices for all commodities. Still, such a strengthening of assumptions A.1 or A.2 does not imply that public production is different from zero in equilibrium.

It is a straightforward exercise to extend the Theorem to the situation with several public firms which follow different pricing rules, as long as public firms produce distinct outputs. On the other hand, two special cases seem to be worth to be pointed out. The first concerns the situation where public outputs are neither available as endowments nor as competitive output. This provides existence in particular for the case with no private endowment and labor as the only competitive input. The second concerns the type of non-convexities allowed for the public production possibility set. Although the situation of positive fixed cost cannot be dealt with in the proof [in A.4 it is assumed that $C(0, w) = 0$ for all w], the case of a technology with typical "lumpy" inputs or output "jumps" could, if a slightly weaker condition than cost minimization is imposed.

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