ON CHILDERN'S QUANTITATIVE CONCEPT OF RATIONAL NUMBER: CONSTRUCT AND ESTIMATE THE SUM

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RESUME

Ce texte est un des nombreux rapports issus du "Rational Number Project" touchant la question de l'élaboration d'un concept quantitatif de nombre rationnel chez les enfants de quatrième et cinquième année. Comme mesure possible de ce concept, cette étude rend compte des performances des enfants dans une tâche d'estimation pour laquelle le temps était limité et dans laquelle ils avaient à construire, à partir de chiffres choisis, une somme de deux nombres rationnels qui devait être aussi près que possible de l'unité. Deux versions de cette tâche furent employées. A partir de l'écart-moyen à l'unité dans trois essais de la tâche, les enfants furent classés comme exécutants forts, moyens ou faibles. Les explications obtenues dans des entrevues individuelles, permirent de caractériser les mécanismes cognitifs exhibés par les exécutants forts comme étant le recours à l'application souple et spontannée des concepts d'ordre des nombres rationnels et d'équivalence de fraction et le recours à un procédé identifiable d'estimation (utilisation d'un point de référence).
Les exécutants faibles furent caractérisés par l'absence manifeste de ces mécanismes cognitifs ou par des manifestations limitées, incorrectes ou grossières de tels mécanismes cognitifs.

A good understanding of fraction size seems important for children to use rational-number concepts in computation and problem solving; however, assessment of this is difficult. This paper considers an estimation task of constructing a two-addend rational-number-sum close to 1 as a measure of the quantitative concept of rational number, and is an elaboration of results presented by Wachsmuth, et al. (1983).

The research literature concerning children's ability to estimate arithmetic computations is sparse and is especially so for rational-number computation. A National Assessment of Educational Progress (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980) found that only 24 and 37%, respectively, of 13- and 17-year-olds selected correctly from among 1, 2, 19, and 21 to estimate 12/13 + 7/8; most frequent responses were 19 and 21. Trafton (1978) identified four estimation processes and indicated that "use of a reference point" promotes reflective thinking. Streefland (1982) investigated the effect of estimation on ratio and proportion tasks; results suggest that such reflective thinking, which can be provoked and developed by the demands for estimating, organizes the problem domain and prestructures the solving process. Payne and Seber (1959) and Buchanan (1978) indicate that the ability to estimate requires early experience with concepts of more, less, between, and equals. Bright (1976) suggests the ability to estimate measurements requires the development of points of reference. Estimation is indispensible to the notion of "quantitative literacy" (Siegel & Goldsmith, 1982). A positive correlation of 0.74 between college students' scores on the School and College Ability Test (SCAT) quantitative ability subtest and an author-constructed test of estimation ability was reported by Levine (1982).

The research literature suggests that children's ability to make good estimates of arithmetical computations is related to their concept of number size. This paper considers a competitive rational-number estimation task of constructing, under time constraints and from a limited set of numerals, a two-addend rational-number sum close to 1. The purposes of this investigation were to (a) gain information about children's accuracy on such a task and (b) gain insights into the cognitive mechanisms exhibited by successful children.

THE STUDY

The present study was conducted by the Rational Number Project during 1982-83. Instruction in a 30-week teaching experiment was based on the multiple-embodiment principle (Dienes, 1971) and included the use of several manipulative materials, representational modes, and rational-number constructs (Behr, et al., 1980). An important aspect was that translations between different modes of representations which subjects frequently were to make during instruction would facilitate the abstraction of rational-number ideas (Lesh, et al., 1980). The study presented in this paper is part of a larger set of studies aimed at assessing children's quantitative notion of (positive) rational number (see Wachsmuth et al., 1983). Subjects were eight children at DeKalb, selected to reflect the full range of ability, and a classrooom size group of 34 homogeneously-grouped middle-ability children at Minneapolis-St. Paul. Both groups were instructed and observed during their fourth and fifth grades. Data from two clinical interviews with the DeKalb children and with eight from Minneapolis form the basis of this study.

Version 1 of the Estimate-the-Sum task consisted of six cards with the numerals 1, 3, 4, 5, 6, 7 and a form board as shown in Figure 1. Version 2 used cards with 11, 3, 4, 5, 6, 7. Version 1 was presented as part of a video-taped interview following 20 weeks of instruction; both were presented after 27 weeks of instruction. Subjects were directed to "put number cards inside the boxes to make fractions so that when you add them the answer is as close to one as possible, but not equal to one." To discourage the use of algorithms, subjects were encouraged to estimate, and a time limit of one minute was imposed on the task.

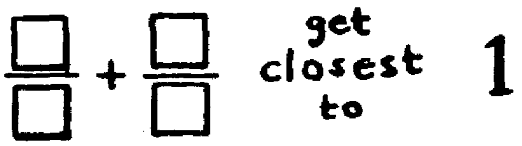

Figure 1

RESULTS

The percentage deviations of constructed responses from 1 varied from 2.38% to 491.71% with an average percentage deviation over all subjects on all tasks of 52.75%. Based on children's explanations, the responses were classified into four categories plus an "other" category. Sample responses exhibiting subjects' thinking on the tasks are given with descriptions of the categories.

CATEGORY ER (Estimate by correct comparison to a Reference point). Response explanations indicate a successful attempt to estimate the constructed rational-number sum by using one-half, 1, or some other self-chosen fraction as point of reference. The spontaneous use of fraction equivalence and rational-number order is evident.

BERT: [Using 1 3 4 5 6 7 constructs 5/6 + 1/7]...well uh,...five ...five-sixths [pointing to 5/6] is one piece away from the unit, and a seventh is just a little bit smaller, so that could fit there (i.e., between 5/6 and 1).

KRISTY: [From 11 3 4 5 6 7 constructs 6/11 + 3/7 and changes to 5/11 + 3/7] ... Well five and a half is half of eleven [pointing to 5/11] and [pointing to 3/7] three and a half is half of seven, so it would be one away from ... (and I changed 6/11 to 5/11) ... because [pointing to 6/11] that would be a little more, that's [pointing to 3/7] less than one (-half) ... I was afraid they'd get exactly one.

BERT: [From 11 3 4 5 6 7 makes 3/6 + 5/11] ... Three-sixths is half a unit and ... if it was five and a-half-elevenths [pointing to 5/11] that would be half; and a half (i.e., one-half-elevenths) would be very thin.

CATEGORY MC (Mental algorithmic Computation). Response explanations indicate that the subject did mental computation to carry out a correct standard algorithm (e.g. common denominator) to determine the actual sum of the generated fractions. The spontaneous use of fraction equivalence and rational-number order is evident.

KRISTY: [Using 1 3 4 5 6 7 makes □/3 + □/4, then changes to 1/3 + 4/5] ... If you find the common denominator, twelve; but ..., and then four times one would be four [explaining the change □/4 to 4/5] , but then three times ... I didn't have a two or anything (among the number cards given and remaining) and I used up my three so ... .

CATEGORY ERI (Estimate by Incorrect, gross, or uncertain comparison to a Reference point). Response explanations indicate that the subject attempted to estimate the constructed rational-number sum by using one-half, 1, or some other self-chosen fraction as a point of reference. Little or constrained understanding of fraction equivalence and rational-number order is evident.

JESSIE: [From 1 3 4 5 6 7 makes 1/3 + 5/6] ... [points to 5/6] that's less than a half and ... if you add wait ... if you add one-third it would be bigger ... [points to 5/6] that ... that's bigger than one-half ... that's (i.e. 1/3) less than one-half ... (and 5/6) is less than half, wait bigger than, less than half ...

MACK: [From 1 3 4 5 6 7 makes 5/6 + 3/4] I just thought about equivalent fractions ... like ... wait ... like you take closest to one you can get, three-fourths 'cause it's only one (i.e. one-fourth) away (from 1), and the same with this one [pointing to 5/6].

CATEGORY MCI (Mental algorithmic Computation based on Incorrect algorithm). Responses indicate that the subject used mental computation based on an incorrect algorithm to compute the actual sum.

TED: [From 1 3 4 5 6 7 makes 5/6 + 4/7] ... Well first I thought, I tried to figure out what would come closest to one and I found out that five-sixths and four-sevenths would come the closest ... 'cause I used the top number ... (If I added them) nine-thirteenths.

JESSIE: [From 1 3 4 5 6 7 makes 3/7 + 4/6] ... Three plus four is seven and that's [pointing to 7 and 6] thirteen ; it would be one-thirteenth close to one.

CATEGORY G (Gross estimate) Response explanations suggest that the subject made a gross estimate of each rational-number addend, but did not make a comparison to a reference point, and did not use fraction equivalence or rational-number ordering.

TED: [From 11 3 4 5 6 7 makes 3/11 + 4/7]... I wanted to use up the little pieces for the top ... then use the highest number of pieces for the bottom ... well, if I ever thought if it was equal, or one's less or greater and stuff, I always have to be greater than the top number.

Responses in Categories ER and MC represent the highest performances as measured by the deviation of the constructed sum from 1. Responses categorized in Category ER (n = 11) had an average deviation of

6.16%, those in Category MC (n = 3) 17.79%,and the percentage across Categories ER and MC was 8.65%. For Category ERI (n = 4), the average deviation was 26.96% and across Categories ER, MC and ERI (n = 18) it was 12.72%. This is contrasted to the average deviation in categories other than ER, MC, and ERI (n = 18), which was 97.65%. (Because of incompleteness, data of two subjects were omitted in this calculation).

DISCUSSION

The subjects had received some instruction on estimation of whole-number sums by rounding, but no formal instruction on estimating the sum of two rational-numbers. They had received extensive experience with rational-number order and fraction equivalence. In some lesson near the assessments relevant for the present study, class activity included one in which children were given a fraction, and they in turn were to give fractions successively closer to the one given.

Among children whose performance on the Estimate-the-Sum task was rated as high, 11 of 14 responses given by five of the 14 subjects considered were in Category ER and the remaining three responses in MC (no response was given by one child in one presentation of the task). That is, children whose performance on this task was rated as high, almost uniformly used estimation; moreover, the estimation process was that of using a "reference point" (Trafton, 1978). They also displayed ability for spontaneous and flexible application of fraction order and equivalence concepts. It appears that a combination of skills in estimation and order and equivalence is important for a good quantitative concept of rational number.

Children whose performance was rated as low on the Estimate-the-Sum task gave no responses in the ER or MC categories but all among MCI, ERI, G, and "other". These responses show an absence of application of order and equivalence concepts and of use of accurate estimation processes such as use of a reference point.

OTHER OBSERVATIONS

While care must be exercised about broad generalizations given the small sample size, it is useful to make additional observations about other cognitive characteristics of children who exhibited Category ER and MC responses as compared to children whose responses fall in other categories. Bert's explanations, in general, exhibit considerable imaginal memory. Exemplified by his responses in Category ER above, there is evidence to suggest that he imagines episodic experiences associated with the manipulative-based instruction. Remarkable is Bert's ability to associate oral symbols for mathematical entities with the imaginal units ("five-sixths is one piece away ..."), exhibiting his ability to translate between ideas expressed via manipulatives and ideas expressed in mathematics symbolism. Kristy's ability to store a sequence of memory units together with tremendous mental symbol-manipulation capability is evident in her Category MC response. She has ability to "preview" an entire algorithm sequence, with her memory needing to deal only with numerical entries of the algorithm; the process is automatic. Kristy and Bert's explanations further demonstrated excellent applications of order and equivalence concepts while some other students displayed an inability to apply these concepts in the estimation task.

In summary, responses which placed subjects in the high performance level on the Estimate-the-Sum task are characterized by a spontaneous and flexible application of fraction order and equivalence concepts and (in most cases) accurate application of an identifiable process of estimation (use of a reference point). Responses which placed subjects in the low performance level are characterized by indicating constrained use of fraction order and equivalence concepts and very uncertain or inaccurate, if any, use of an estimation process. Thus, a high level of understanding, and applicability, of fraction order and equivalence appears necessary for an ability to give estimates of rational numbers and rational-number sums.

As was mentioned, Streefland (1982, p. 197) suggested that estimation can "elicit a global orientation on the problem set, which organises the problem domain and prestructures the solving procedure." We see this prestructuring behavior in Bert and Kristy (see Category ER). This ability to see a global organization of a problem domain was observed (Behr, Wachsmuth, Post, and Lesh, 1983), in children who were able to use manipulative aids flexibly, translate between different aids, and display thinking which had progressed from manipulative dependent to manipulative independent. These issues -- estimation, quantitative concept of number, flexibility of thought, and translation between representational modes -- appear to be interrelated and demand the attention of researchers.

References

Behr, M. J., Post, T. R., Silver, E. A., & Mierkiewicz, D. B. Theoretical foundations for instructional research of rational numbers. In R. Karplus (Ed.), Proceedings of the Fourth International Conference for the Psychology of Mathematics Education, Berkeley, California, 1980.

Behr, M. J., Wachsmuth, I., Post, T. R., & Lesh, R. Order and equivalence of fractions: Concept development among 9-year olds. Manuscript submitted for publication, 1983.

Buchanan, A. D. Estimation as an essential mathematics skill. ERIC DOC. No. ED. 167 385.

Bright, G. W. Estimation as part of learning to measure. In D. Nelson & R. E. Reys (Eds.), Measurement in School Mathematics. Reston, VA.: NCTM, 1976.

Carpenter, T. P., Corbitt, M. K., Kepner, H., Lindquist, M. M., & Reys, R. E. Results and implications of the second NAEP mathematics assessment: Elementary school. Arithmetic Teacher, 1980, 27(8), 10-12, 44-47.

Dienes, Z. P. Building up Mathematics (Second Edition). London: Hutchinson Educational Ltd., 1971.

Lesh, R., Landau, M. & Hamilton, E. Rational number ideas and the role of representational systems. In R. Karplus (Ed.), Proceedings of the Fourth International Conference for the Psychology of Mathematics Education, Berkeley, CA., 1980.

Levine, D. R. Strategy use and estimation ability of college students. Journal for Research in Mathematics Education, 1982, 13, 350-359.

Payne, J. P. & Seber, R. C. Measurement and approximation. In The Growth of Mathematical Ideas Grades K-12. Reston, VA.: NCTM, 1959.

Siegel, A. W., Goldsmith, L. T. & Madson, C. R. Skill in estimation problems of extent and numerosity. Journal for Research in Mathematics Education, 1982, 13(3), 211-231.

Streefland, L. The role of rough estimation in learning ratio and proportion - an exploratory study. In A. Vermandel (Ed.), Proceedings of the Sixth International Conference for the Psychology of Mathematics Education, Antwerp, Belgium, 1982.

Trafton, D. R. Estimation and mental arithmetic: Important components of computation. In M. N. Suydam & R. E. Reys (Eds.), Developing computational skills. Reston, VA.: NCTM, 1978.

Wachsmuth, I., Behr, M. J. & Post, T. R. Children's quantitative notion of rational number. Paper presented at the 1983 Annual Meeting of the American Educational Research Association, Montreal, Canada, April, 1983.