# Perturbation and Interlace Theorems for the Unitary Eigenvalue Problem

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ABSTRACT

Two aspects of the perturbation problem for the eigenvalues of a unitary matrix $U$ are treated. Firstly, analogues of the Hoffman-Wielandt theorem and a Weyl-type theorem proved by Bhatia and Davis are derived, which are based on a different measure of the distance of spectra. Using a suitable parametrization of the unit circle by an angle, the new results are called *tangent theorems*, in contrast to the first-mentioned well-known results, which are *sine theorems*. Moreover, we illuminate the unknown minimizing permutations in the above Weyl-type theorems. With respect to their angles the eigenvalues of $U$ and $\tilde{U}$ (the perturbed matrix) are naturally ordered on the unit circle counterclockwise, after a point is cut on the unit circle. We prove a well-known open conjecture; there exists a cutting point such that the Weyl-type theorems, both sine and tangent, are true when the ordered eigenvalues of $U$ and $\tilde{U}$ are paired with each other. Secondly, the Cauchy interlacing theorem for Hermitian matrices is generalized. It is shown that certain modified principal submatrices of $U$, called the modified $k$th leading principal submatrices, have the property that their eigenvalues interlace those of $U$. Finally we discuss block reflectors, appearing in the description of the modified principal submatrices, and generalize a result of Schreiber and Parlett.

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## 1. INTRODUCTION

In recent years, numerical methods for the unitary eigenvalue problem such as the $QR$ methods [12, 25], the divide-and-conquer method [7, 26], the bisection method [13], and some special methods for the real orthogonal eigenvalue problem [2, 3, 17] have been developed. Applications, e.g. in signal processing [16, 30, 33], in Gaussian quadrature on the unit circle [24], and in trigonometric approximations [32], have led to considerable interest in these methods. In this note we give perturbation and interlacing theorems which are required by the numerical methods for the unitary eigenvalue problem.

Let us first describe the known perturbation results for eigenvalues of unitary matrices. Let $U$ and $\tilde{U}$ be two $n \times n$ unitary matrices with spectra $\operatorname{Eig} U = \{\lambda_j\}$ and $\operatorname{Eig} \tilde{U} = \{\tilde{\lambda}_j\}$ respectively. The following distances between the spectra of $U$ and $\tilde{U}$ were considered in [10, 11]:

$$d_\mu(\operatorname{Eig} U, \operatorname{Eig} \tilde{U}) = \min_P \|\Lambda - P^T \tilde{\Lambda} P\|_\mu, \qquad \mu = 2, F, \tag{1.1}$$

where $\Lambda = \operatorname{diag}(\lambda_j)$, $\tilde{\Lambda} = \operatorname{diag}(\tilde{\lambda}_j)$, $P$ runs over all permutation matrices, and $\| \ \|_2$, $\| \ \|_F$ denote the spectral and Frobenius norms. By the Hoffman-Wielandt theorem [27]

$$d_F(\operatorname{Eig} U, \operatorname{Eig} \tilde{U}) \leqslant \|U - \tilde{U}\|_F, \tag{1.2}$$

and more recently Bhatia and Davis have shown the corresponding result for the spectral norm [10]:

$$d_2(\operatorname{Eig} U, \operatorname{Eig} \tilde{U}) \leqslant \|U - \tilde{U}\|_2. \tag{1.3}$$

Here we will study another measure for the "distance" of the spectra of $U$ and $\tilde{U}$, namely the relative error. More specifically, we define

$$\tilde{d}_\mu(\operatorname{Eig} U, \operatorname{Eig} \tilde{U}) = \min_P \left\|(\Lambda + P^T \tilde{\Lambda} P)^{-1}(\Lambda - P^T \tilde{\Lambda} P)\right\|_\mu, \qquad \mu = 2, F. \tag{1.4}$$

For example, $\tilde{d}_2(\operatorname{Eig} U, \operatorname{Eig} \tilde{U}) \leqslant \epsilon$ means that there exists a minimizing permutation $\pi$ of $\{1, \dots, n\}$ such that

$$\left| \frac{1 - \lambda_j^{-1} \tilde{\lambda}_{\pi(j)}}{1 + \lambda_j^{-1} \tilde{\lambda}_{\pi(j)}} \right| \leqslant \epsilon, \qquad j = 1, \dots, n.$$

We will prove in Section 3 the following bounds:

$$\tilde{d}_F(\operatorname{Eig} U, \operatorname{Eig} \tilde{U}) \leqslant \left\| (U + \tilde{U})^{-1}(U - \tilde{U}) \right\|_F = \left\| C(U^H \tilde{U}) \right\|_F \tag{1.5}$$

and

$$\tilde{d}_2(\operatorname{Eig} U, \operatorname{Eig} \tilde{U}) \leqslant \left\| (U + \tilde{U})^{-1}(U - \tilde{U}) \right\|_2 = \left\| C(U^H \tilde{U}) \right\|_2. \tag{1.6}$$

Here

$$C(U) = i(I + U)^{-1}(I - U) \tag{1.7}$$

is the Cayley-transformation of $U$ (where $-1 \notin \operatorname{Eig} U$) mapping unitary matrices into Hermitian matrices.

We can interpret (1.1) and (1.4) in terms of the angles of the eigenvalues defined in (1.9) below. The Cayley transformation

$$x = i(1 + \lambda)^{-1}(1 - \lambda), \qquad |\lambda| = 1, \tag{1.8}$$

maps the unit circle one-to-one onto the extended real line. Defining

$$\theta_\lambda = \arctan\left[ i(1 + \lambda)^{-1}(1 - \lambda) \right], \tag{1.9}$$

each $\lambda$ on the unit circle corresponds to an angle $\theta_\lambda$, $-\pi/2 \leqslant \theta_\lambda < \pi/2$ (see Figure 1). Equation (1.4) is based then on the distance function

$$\tilde{d}(\lambda, \tilde{\lambda}) = \left| \tan(\theta_\lambda - \theta_{\tilde{\lambda}}) \right| = \left| \frac{\lambda - \tilde{\lambda}}{\lambda + \tilde{\lambda}} \right|. \tag{1.10}$$

Also, the usual bound $d(\lambda, \tilde{\lambda}) = |\lambda - \tilde{\lambda}|$ can be expressed in terms of the corresponding angles $\theta_\lambda$ and $\theta_{\tilde{\lambda}}$:

$$\left| \sin(\theta_\lambda - \lambda_{\tilde{\lambda}}) \right| = \tfrac{1}{2} |\lambda - \tilde{\lambda}|. \tag{1.11}$$

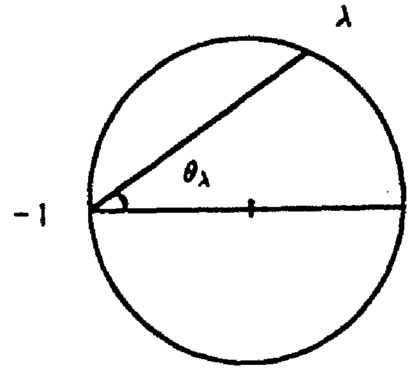


FIG. 1.

Introducing the angles of the eigenvalues

$$\theta_j = \theta_{\lambda_j}, \quad \tilde{\theta}_j = \theta_{\tilde{\lambda}_j}, \qquad j = 1, \dots, n,$$

and the standard ordering

$$-\frac{\pi}{2} \leqslant \theta_1 \leqslant \cdots \leqslant \theta_n < \frac{\pi}{2}, \qquad -\frac{\pi}{2} \leqslant \tilde{\theta}_1 \leqslant \cdots \leqslant \tilde{\theta}_n < \frac{\pi}{2},$$

the perturbation bounds (1.2)(1.3) and (1.5)(1.6) can then be expressed in the following form: There are permutations $\pi_k$, $k = 1, \dots, 4$, of $\{1, \dots, n\}$ such that

$$\left( \sum_j \sin^2\left(\theta_j - \tilde{\theta}_{\pi_1(j)}\right) \right)^{1/2} \leqslant \frac{\|U - \tilde{U}\|_F}{2}, \tag{1.12}$$

$$\max_j \left| \sin\left(\theta_j - \tilde{\theta}_{\pi_2(j)}\right) \right| \leqslant \frac{\|U - \tilde{U}\|_2}{2}, \tag{1.13}$$

$$\left( \sum_j \tan^2\left(\theta_j - \tilde{\theta}_{\pi_3(j)}\right) \right)^{1/2} \leqslant \left\| (U + \tilde{U})^{-1} (U - \tilde{U}) \right\|_F, \tag{1.14}$$

$$\max_j \left| \tan\left(\theta_j - \tilde{\theta}_{\pi_4(j)}\right) \right| \leqslant \left\| (U + \tilde{U})^{-1} (U - \tilde{U}) \right\|_2. \tag{1.15}$$

A natural question is whether the permutations $\pi_k$, $k = 1, \dots, 4$, can be chosen to be the identity. For the Weyl-type inequalities (1.13) and (1.15), as we will show in Section 4, this is true in a slightly weaker sense. To do this we define

$$\theta_j(\xi) = \arctan\left(i\frac{\xi+\lambda_j}{\xi-\lambda_j}\right), \quad \tilde{\theta}_j(\xi) = \arctan\left(i\frac{\xi+\tilde{\lambda}_j}{\xi-\tilde{\lambda}_j}\right), \qquad j = 1,\dots,n,$$

as angles of the eigenvalues of $U$ and $\tilde{U}$ according to a new cutting point $\xi$ on the unit circle, satisfying

$$-\frac{\pi}{2} \leqslant \theta_1(\xi) \leqslant \cdots \leqslant \theta_n(\xi) < \frac{\pi}{2},$$

$$-\frac{\pi}{2} \leqslant \tilde{\theta}_1(\xi) \leqslant \cdots \leqslant \tilde{\theta}_n(\xi) < \frac{\pi}{2}.$$

We will prove that there exists a cutting point $\xi$ on the unit circle such that

$$\max_j \left|\sin\left[\theta_j(\xi) - \tilde{\theta}_j(\xi)\right]\right| \leqslant \frac{\|U-\tilde{U}\|}{2}, \tag{1.16}$$

$$\max_j \left|\tan\left[\theta_j(\xi) - \tilde{\theta}_j(\xi)\right]\right| \leqslant \left\|(U+\tilde{U})^{-1}(U-\tilde{U})\right\|. \tag{1.17}$$

Another topic that will be discussed, in Section 5, is interlacing. Some earlier results on this topic are restricted to rank-1 perturbed unitary matrices [4, 21]. For Hermitian matrices the eigenvalues of a principal submatrix interlace those of the complete matrix, which is known as the *Cauchy interlacing theorem*. Here we show that such a result holds also for unitary matrices, if we define "principal submatrices" appropriately and define "interlacing" in an obvious manner.

Given a unitary matrix $U$ with $-1 \notin \operatorname{Eig} U$, it is shown that for any $k \leqslant n$, there exists a unique $k \times k$ unitary matrix $U_k$ such that

$$\operatorname{rank}\left[\begin{pmatrix} U_k^H & 0 \\ 0 & -I_{n-k} \end{pmatrix} U - I\right] = n - k.$$

In other words, there exists a unique decomposition

$$U = \begin{pmatrix} U_k & 0 \\ 0 & -I_{n-k} \end{pmatrix} G, \qquad \operatorname{rank}(G - I) = n - k. \tag{1.18}$$

We call $U_k$ the $k$th modified leading principal submatrix, and show that its eigenvalues interlace those of $U$. More exactly, if $U$ has eigenvalues $\lambda_1, \dots, \lambda_n$ ordered in such a way that the corresponding angles $\{\theta_j\}$ satisfy

$$-\frac{\pi}{2} < \theta_1 \leqslant \theta_2 \leqslant \cdots \leqslant \theta_n < \frac{\pi}{2},$$

and similarly for the angles $\{\tau_j\}$ corresponding to the eigenvalues $\{\mu_j\}$ of $U_k$, then

$$\theta_j \leqslant \tau_j \leqslant \theta_{n+j-k}, \qquad 1 \leqslant j \leqslant k. \tag{1.19}$$

Our interlacing result uses a different description of the modified principal submatrix in (1.18) via the inverse Cayley-transform. The matrix $G$ in (1.18) is actually a block reflector [8, 34]. It is of the form $G = I - XDX^H$, where $X$ is an $n \times (n-k)$ matrix with orthogonal columns and $D$ is a $(n-k) \times (n-k)$ diagonal matrix satisfying $(I - D)(I - D^H) = I$. One important application of this block reflector is that for any two $n \times k$ matrices $E$ and $F$ satisfying $E^H E = F^H F$, there exists a block reflector $G$ with $\operatorname{rank}(I - G) \leqslant k$ such that $E = G^H F$.

## 2. PERTURBATION THEOREMS. I. SINE THEOREMS

In this section we recall the Hoffman-Wielandt theorem (see [27], where more generally the proof for normal matrices is given) as Theorem 2.1, and the Weyl-type theorem by Bhatia and Davis (see [10, 11], but the result is wrong if "unitary" is replaced by "normal") as Theorem 2.2. For comparison with the tangent formulas in the next section we have formulated them in terms of sines according to (1.11). The perturbed matrix $\tilde{U}$ is conveniently denoted by $\tilde{U} = US$, so the bounds are $\|U - \tilde{U}\|_\mu = \|I - S\|_\mu$, $\mu = F, 2$.

THEOREM 2.1. *Suppose that $\{\theta_j\}$ and $\{\tilde{\theta}_j\}$ are the angles corresponding to $\{\lambda_j\}$ and $\{\tilde{\lambda}_j\}$ with respect to the cutting point $-1$. Then there exists a permutation $\pi_1$ of $\{1, \dots, n\}$ such that*

$$\sum_{j=1}^{n} \left| \sin\left(\theta_j - \tilde{\theta}_{\pi_1(j)}\right) \right|^2 \leqslant \frac{\|I - S\|_F^2}{4}. \tag{2.1}$$

THEOREM 2.2. *Suppose that* $\{\theta_j\}$ *and* $\{\tilde{\theta}_j\}$ *are the angles corresponding to* $\{\lambda_j\}$ *and* $\{\tilde{\lambda}_j\}$ *with respect to the cutting point* $-1$. *Then there exists a permutation* $\pi_2$ *of* $\{1, \dots, n\}$ *such that*

$$\max_j \left|\sin\left(\theta_j - \tilde{\theta}_{\pi_2(j)}\right)\right| \leqslant \frac{\|I - S\|_2}{2}. \tag{2.2}$$

## 3. PERTURBATION THEOREMS. II. TANGENT THEOREMS

In this section, we shall give the perturbation theorems for the tangents of the angles, which can be regarded as relative errors of eigenvalues of U, in contrast to the absolute errors of the sine theorems. To do this we first prove the following lemma.

LEMMA 3.1. *Let* $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ *with* $|\lambda_j| = 1$, $j = 1, \dots, n$, *let* $\tilde{\Lambda} = \operatorname{diag}(\tilde{\lambda}_j)$ *with* $|\tilde{\lambda}_j| = 1$, $j = 1, \dots, n$, *and let*

$$f(Q) = \left\|\left(\Lambda + Q^H\tilde{\Lambda}Q\right)^{-1}\left(\Lambda - Q^H\tilde{\Lambda}Q\right)\right\|_F. \tag{3.1}$$

*Then* $\min\{f(Q) : Q \text{ unitary}\} = f(P)$ *for a suitable permutation* $P$.

*Proof*. It suffices to prove the statement for the case that $\lambda_i \neq \lambda_j$ and $\tilde{\lambda}_i \neq \tilde{\lambda}_j$ for $i \neq j$. By compactness there exists a unitary $Q_0$ minimizing $f(Q)$. Define for a given Hermitian matrix $H$

$$g(t) = f^2\left(Q_0 e^{iHt}\right). \tag{3.2}$$

As $e^{itH}$ is unitary, one has $g(t) \geqslant g(0)$, and hence its derivative $g'(t)$ vanishes at 0: $g'(0) = 0$. A tedious calculation gives

$$g(t) = 4\operatorname{tr}\left\{\left[2I + V(t) + V^H(t)\right]^{-1}\right\} - n,$$

where $V(t) = \Lambda^H e^{-itH} Q_0^H \tilde{\Lambda} Q_0 e^{itH}$. Introducing $V_0 = V(0)$ and

$A = 2I + V_0 + V_0^H$, this leads to

$$\begin{aligned}0 = g'(0) &= -4\,\mathrm{tr}\{A^{-2}[\Lambda^H(-iH)\Lambda V_0 \\ &\qquad + V_0(iH) + V_0^H\Lambda^H(iH)\Lambda + (-iH)V_0^H]\} \\ &= -4i\,\mathrm{tr}[H(-\Lambda V_0 A^{-2}\Lambda^H + A^{-2}V_0 + \Lambda A^{-2}V_0^H\Lambda^H - V_0^H A^{-2})],\end{aligned}$$

whence, by Lemma 3.2, we have

$$-\Lambda V_0 A^{-2}\Lambda^H + A^{-2}V_0 + \Lambda A^{-2}V_0^H\Lambda^H - V_0^H A^{-2} = 0. \tag{3.3}$$

Setting $A^{-2}V_0 = W = (w_{ij})$ and observing that $A^{-2}V_0 = V_0 A^{-2}$, we have by (3.3)

$$-\Lambda W\Lambda^H + W + \Lambda W^H\Lambda^H - W^H = 0,$$

or equivalently

$$\left(1 - \lambda_i\bar{\lambda}_j\right)\left(w_{ij} - \bar{w}_{ji}\right) = 0, \qquad 0 \leqslant i, j \leqslant n.$$

By the assumption above, the numbers $1 - \lambda_i\bar{\lambda}_j$ are nonzero for $i \neq j$, and hence $\Delta = W - W^H = (2I + V_0 + V_0^H)^{-2}(V_0 - V_0^H)$ is diagonal. Now we prove that $V_0$ is diagonal. We have a spectral decomposition of $V_0$ of the form $V_0 = XD_0X^H$, where $D_0 = \mathrm{diag}(d_1 I_{k_1}, \dots, d_r I_{k_r})$ and $d_i \neq d_j$ for $i \neq j$ and $X$ is unitary. Hence

$$\Delta = X\left(2I + D_0 + D_0^H\right)^{-2}\left(D_0 - D_0^H\right)X^H.$$

But as $\Delta$ is diagonal, we have also for a suitable permutation $P_0$

$$\Delta = P_0^T\left(2I + D_0 + D_0^H\right)^{-2}\left(D_0 - D_0^H\right)P_0. \tag{3.4}$$

This shows that $P_0 X$ commutes with the diagonal matrix

$$\tilde{\Delta} = \left(2I + D_0 + D_0^H\right)^{-2}\left(D_0 - D_0^H\right) = \mathrm{diag}(\Delta_1 I_{k_1}, \dots, \Delta_r I_{k_r}).$$

here $\Delta_i \neq \Delta_j$ for $i \neq j$, which can be easily verified. It follows that

$P_0 X = \text{diag}(X_1, \dots, X_r)$ with the $k_i \times k_i$ unitary matrices $X_i$, $i = 1, \dots, r$. Thus $(P_0 X) D_0 (P_0 X)^H = D_0$ and $V_0 = P_0^T D_0 P_0$ is diagonal. But $V_0 = \Lambda Q_0^H \tilde{\Lambda} Q_0$, and as $\lambda_i$ are different, $Q_0$ is a permutation. ∎

LEMMA 3.2. *Given a square matrix $X$. If for any Hermitian matrix $H$ one has* $\text{tr}(HX) = 0$, *then* $X = 0$.

*Proof*. Consider two special choices of $H_1 = X + X^H$ and $H_2 = i(X^H - X)$. Then from the condition of Lemma 3.2, $\text{tr}[(X + X^H)X] = 0$ and $\text{tr}[(X^H - X)X] = 0$. The sum of these two traces is $2\,\text{tr}(X^H X) = 0$. So $X = 0$. ∎

It is clear that the value of $f(P)$ can be written as a sum of tangents with the bound $\|(I + S)^{-1}(I - S)\|_F = \|(\Lambda + Q^H \tilde{\Lambda} Q)^{-1}(\Lambda - Q^H \tilde{\Lambda} Q)\|_F = f(Q)$, where $U = Q_1 \Lambda Q_1^H$, $\tilde{U} = Q_2 \tilde{\Lambda} Q_2^H$, and $Q = Q_2^H Q_1$. Thus Lemma 3.1 implies the following theorem.

THEOREM 3.3. *Under the assumptions of Theorem* 2.1 *and* (1.10), *there exists a permutation* $\pi_3$ *of* $\{1, \dots, n\}$ *such that*

$$\sum_{j=1}^{n} \left| \tan\left(\theta_j - \tilde{\theta}_{\pi_3(j)}\right) \right|^2 \leqslant \left\| (I + S)^{-1} (I - S) \right\|_F^2. \tag{3.5}$$

For the bound $\|(I + S)^{-1}(I - S)\|_2$, we have the following theorem, which is directly obtained from Theorem 2.2.

THEOREM 3.4. *Under the assumptions of Theorem* 2.1 *and* (1.10), *there exists a permutation* $\pi_4$ *of* $\{1, \dots, n\}$ *such that*

$$\max_j \left| \tan\left(\theta_j - \tilde{\theta}_{\pi_4(j)}\right) \right| \leqslant \left\| (I + S)^{-1} (I - S) \right\|_2. \tag{3.6}$$

*Proof*. Let $\{\beta_j\}$ be the eigenvalues of $S$, and their corresponding angles be $\{\eta_j\}$; then Theorem 2.2 shows that

$$\left| \sin\left(\theta_{i_0} - \tilde{\theta}_{\pi_2(i_0)}\right) \right| \leqslant |\sin \eta_{j_0}|,$$

where $|\sin(\theta_i - \tilde{\theta}_{\pi_2(i)})|$ attains its maximum at $i_0$, and $|\sin \eta_j|$ at $j_0$. Now we have to prove

$$\left| \tan\left(\theta_{i_0} - \tilde{\theta}_{\pi_2(i_0)}\right) \right| \leqslant |\tan \eta_{j_0}|.$$

This holds because $\tan\theta = x/(1-x^2)^{1/2}$, where $x = \sin\theta$, is an increasing function in $x$. By the same argument

$$\left|\tan\left(\theta_{i_0} - \tilde{\theta}_{\pi_2(i_0)}\right)\right| = \max_i \left|\tan\left(\theta_i - \tilde{\theta}_{\pi_2(i)}\right)\right|,$$

$$|\tan\eta_{j_0}| = \|(I+S)^{-1}(I-S)\|_2.$$

So we have (3.6) with $\pi_4 = \pi_2$. ∎

## 4. PERTURBATION THEOREMS. III. ORDERED EIGENVALUES

The eigenvalues $\{\lambda_j\}$, $\{\tilde{\lambda}_j\}$ of Hermitian matrices $A$, $\tilde{A}$ can be ordered in a natural way:

$$\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n, \qquad \tilde{\lambda}_1 \leqslant \tilde{\lambda}_2 \leqslant \cdots \leqslant \tilde{\lambda}_n.$$

Moreover this ordering leads to optimal matchings in the following sense:

$$\max_j |\lambda_j - \tilde{\lambda}_j| \leqslant \max_j |\lambda_j - \tilde{\lambda}_{\pi(j)}|,$$

$$\sum_{j=1}^n |\lambda_j - \tilde{\lambda}_j|^2 \leqslant \sum_{j=1}^n |\lambda_j - \tilde{\lambda}_{\pi(j)}|^2,$$

for any permutation $\pi$ of $\{1, 2, \ldots, n\}$, and hence to sharper versions of the Weyl theorem and the Hoffman-Wielandt theorem for Hermitian matrices. We will show that a result analogous to the Weyl theorem holds for unitary matrices (see Theorem 4.3).

Let us consider first the order of any two complex numbers on the unit circle, $\lambda_1$ and $\lambda_2$. After cutting the unit circle at $\xi$, we define the angles of $\lambda_1$ and $\lambda_2$ by

$$\theta_j(\xi) = \arctan\left(i\frac{\xi + \lambda_j}{\xi - \lambda_j}\right), \qquad j = 1, 2; \tag{4.1}$$

and define $\lambda_1 \leqslant \lambda_2$, if $\tan\theta_1(\xi) \leqslant \tan\theta_2(\xi)$. This means that when moving around the unit circle counterclockwise from the point $\xi$ to the point $\xi$, one first reaches $\lambda_1$, then $\lambda_2$. In this way we assume that the eigenvalues of $U$ and $\tilde{U}$ have the order

$$\lambda_1(\xi) \leqslant \lambda_2(\xi) \leqslant \cdots \leqslant \lambda_n(\xi), \tilde{\lambda}_1(\xi) \leqslant \tilde{\lambda}_2(\xi) \leqslant \cdots \leqslant \tilde{\lambda}_n(\xi) \quad (4.2)$$

with respect to their angles

$$\begin{aligned} -\frac{\pi}{2} \leqslant \theta_1(\xi) \leqslant \theta_2(\xi) \leqslant \cdots \leqslant \theta_n(\xi) < \frac{\pi}{2}, \\ -\frac{\pi}{2} \leqslant \tilde{\theta}_1(\xi) \leqslant \tilde{\theta}_2(\xi) \leqslant \cdots \leqslant \tilde{\theta}_n(\xi) < \frac{\pi}{2}. \end{aligned} \quad (4.3)$$

(see Figure 2).

Notice that $\{\lambda_j(\xi)\}$ is the same as $\{\lambda_j\}$ except for the ordering. For a different cutting point the orders of the eigenvalues are only changed cyclically. Moreover, for different cutting points the inequalities (2.2) and (3.6) hold, as by a direct calculation

$$\theta_1(\xi) - \theta_2(\xi) = \arctan\left((-i)\frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}\right)$$

is a constant with respect to $\xi$.

The following theorem is a natural extension of the Weyl theorem for the symmetric eigenvalue problem. Before proving that, we need two lemmas. Throughout this section we use the notation $(\lambda_1, \lambda_2)$ to denote the open arc from the point $\lambda_1$ to the point $\lambda_2$ on the unit circle counterclockwise.

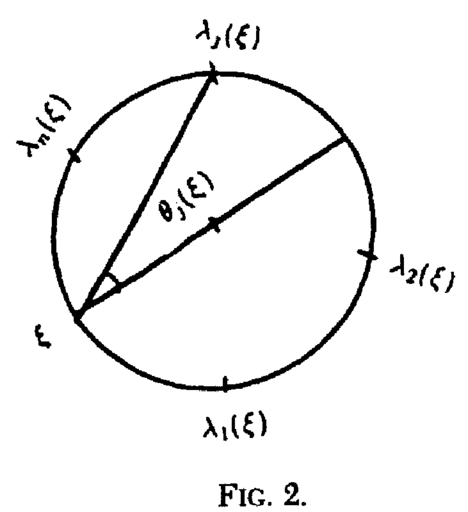


FIG. 2.

LEMMA 4.1. *Suppose that* $-1$ *is the cutting point of the unit circle and* $a$, $\tilde{a}$, $b$, *and* $\tilde{b}$ *are complex numbers on the unit circle such that*

$$a < b, \qquad \tilde{a} < \tilde{b}.$$

*If in addition* $a < \tilde{a}$ *and the arcs connecting* $a, \tilde{a}, \tilde{b}$ *and* $\tilde{a}, \tilde{b}, b$ *lie each on a semicircle, then*

$$\max\{|a - \tilde{a}|, |b - \tilde{b}|\} \leqslant \max\{|a - \tilde{b}|, |b - \tilde{a}|\}.$$

*Proof.* Let $d_1 = \max\{|a - \tilde{b}|, |b - \tilde{a}|\}$. Under the ordering of $a < b$ and $a < \tilde{a} < \tilde{b}$, there are three possible situations for $a$, $b$, $\tilde{a}$, and $\tilde{b}$:

(1) $a < b < \tilde{a} < \tilde{b}$,
(2) $a < \tilde{a} < b < \tilde{b}$,
(3) $a < \tilde{a} < \tilde{b} < b$,

which are shown in Figure 3. The condition that $a, \tilde{a}, \tilde{b}$ are on the same semicircle guarantees that the inequality

$$\max\{|a - \tilde{a}|, |b - \tilde{b}|\} \leqslant d_1$$

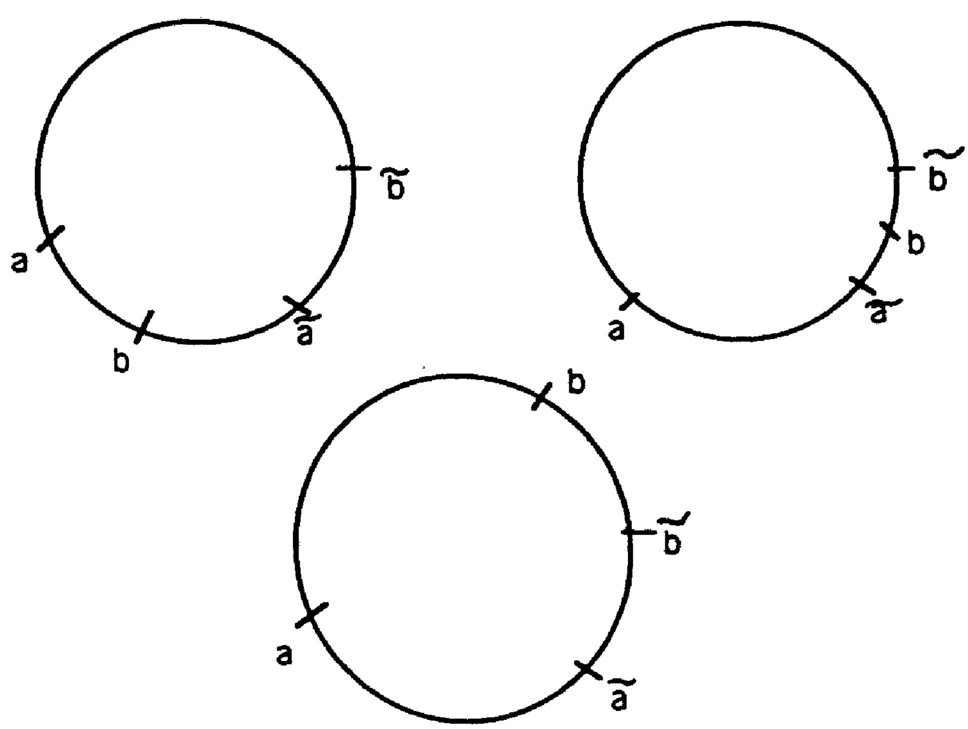


FIG. 3.

holds in the first two cases, and the condition that $\tilde{a}, \tilde{b}, b$ are on the same semicircle guarantees that the above inequality holds in the third case. ∎

We remark that the cutting point $-1$ in the above lemma is only used to make the presentation easy. In fact Lemma 4.1 shows that for two pairs $a, b$ and $\tilde{a}, \tilde{b}$ the minimum of the maximal norms of differences occurs when $a$ or $b$ chooses its nearest point of $\tilde{a}$ and $\tilde{b}$ as a partner.

LEMMA 4.2. *Let $\lambda_j$ and $\tilde{\lambda}_j$ be counterclockwise ordered on the unit circle with respect to the cutting point $-1$. Then there exists an integer $t$ such that*

$$\max_j |\lambda_j - \tilde{\lambda}_{t+j-1}| = \min_\pi \max_j |\lambda_j - \tilde{\lambda}_{\pi(j)}|, \tag{4.4}$$

*where $\tilde{\lambda}_{t+j-1}$ are naturally taken as $\tilde{\lambda}_{t+j-1-n}$ when $t+j-1>n$.*

*Proof.* Let $\pi_2$ be a minimizing permutation of (4.4), and

$$d = |\lambda_1 - \tilde{\lambda}_{\pi_2(1)}| = \max_j |\lambda_j - \tilde{\lambda}_{\pi_2(j)}|. \tag{4.5}$$

Without loss of generality we assume that $\lambda_j$ and $\tilde{\lambda}_j$ are distinct and $|\lambda_j - \tilde{\lambda}_{\pi_2(j)}| < d$ for $j \neq 1$. The integer $t$ to be chosen in this case is just $t = \pi_2(1)$.

It is no restriction to assume that $\lambda_1 < \tilde{\lambda}_t$. In the expression (4.5), each eigenvalue $\lambda_j$ is paired with $\tilde{\lambda}_{\pi_2(j)}$, and $|\lambda_j - \tilde{\lambda}_{\pi_2(j)}| < d$ for $j \neq 1$. As $\lambda_1$ is paired with $\tilde{\lambda}_t$, let us begin with the pairs $\lambda_2$ with $\tilde{\lambda}_{\pi_2(2)}$. The idea is to reorder $\tilde{\lambda}_j$ without changing the minimal distance $d$ in (4.5). In order to use Lemma 4.1, we consider another pair: $\lambda_s$ with $\tilde{\lambda}_{\pi_2(s)}$, where $\pi_2(s) = t+1$ [if $s = 2$, we begin with the pairs $\lambda_3$ with $\tilde{\lambda}_{\pi_2(3)}$ and $\lambda_s$ with $\tilde{\lambda}_{\pi_2(s)}$, where $\pi_2(s) = t+2$]. We have to discuss two cases.

(1) In the case of $\lambda_2 < \tilde{\lambda}_{t+1}$, we have $\tilde{\lambda}_{t+1} < \tilde{\lambda}_{\pi_2(2)}$, as otherwise the minimal distance (4.5) is reduced by exchanging $\pi_2(1)$ and $\pi_2(2)$ in $\pi_2$ by Lemma 4.1. It follows that $\lambda_2 < \tilde{\lambda}_{t+1} < \tilde{\lambda}_{\pi_2(2)}$ and $\lambda_2 < \lambda_s$, and the condition that $\lambda_2, \tilde{\lambda}_{t+1}, \tilde{\lambda}_{\pi_2(2)}$ are on the same semicircle and $\tilde{\lambda}_{t+1}, \tilde{\lambda}_{\pi_2(2)}, \lambda_s$ are on the same semicircle follows by the same argument, since $d$ is the minimal distance. Thus, from Lemma 4.1, $\pi_2(2), \pi_2(s)$ can be exchanged in $\pi_2$ without changing the minimal distance $d$.

(2) In the case of $\tilde{\lambda}_{t+1} < \lambda_2$, we have directly $\tilde{\lambda}_{t+1} < \lambda_2 < \lambda_s$ and $\tilde{\lambda}_{t+1} < \tilde{\lambda}_{\pi_2(2)}$. Thus, from Lemma 4.1 we can again arrange a new permutation such that (4.5) is true and $\lambda_2$ is paired with $\tilde{\lambda}_{t+1}$.

After the pair $\lambda_1$ with $\tilde{\lambda}_t$ and $\lambda_2$ with $\tilde{\lambda}_{t+1}$ have been fixed in the expression (4.5) for the changed permutation $\pi_2$, in the same way we can exchange $\pi_2(3)$ with $\pi_2(s)$ without changing $d$, where $t + 2 = \pi_2(s)$. This process continues until $\pi_2$ is changed into $\{t, t+1, \dots, n, 1, \dots, t-1\}$. ■

THEOREM 4.3. *There exists a cutting point $\xi$ by which $\{\lambda_j(\xi)\}$ and $\{\tilde{\lambda}_j(\xi)\}$ are naturally ordered on the unit circle in the sense of* (4.2) *and* (4.3) *so that*

$$\max_j \left|\sin\left[\theta_j(\xi) - \tilde{\theta}_j(\xi)\right]\right| \leqslant \frac{\|I - S\|_2}{2} \tag{4.6}$$

and

$$\max_j \left|\tan\left[\theta_j(\xi) - \tilde{\theta}_j(\xi)\right]\right| \leqslant \left\|(I+S)^{-1}(I-S)\right\|_2. \tag{4.7}$$

*Proof.* We only need to prove the sine inequality (4.4); then the tangent inequality (4.5) becomes trivial, as Theorem 3.4 follows from Theorem 2.2. We assume for convenience that all $\lambda_j$ and $\tilde{\lambda}_j$ are distinct and all $|\lambda_i - \tilde{\lambda}_j|$ are distinct too.

By Lemma 4.2, $\lambda_j$ is paired with $\tilde{\lambda}_{t+j-1}$, $j = 1, \dots, n$, and $d = |\lambda_1 - \tilde{\lambda}_t|$. Consider $(\lambda_1, \tilde{\lambda}_t)$, and assume that $i_1$ eigenvalues, $\lambda_2, \dots, \lambda_{i_1+1}$, are in this interval. If $i_1$ is zero, we take the cutting point $\xi$ as the point just after $\tilde{\lambda}_t$ on the unit circle; with this cutting point $\xi$, $\lambda_j = \lambda_{j-1}(\xi)$ is paired with $\tilde{\lambda}_{t+j-1}(\xi) = \tilde{\lambda}_{j-1}(\xi)$, and $\lambda_1 = \lambda_n(\xi)$ with $\tilde{\lambda}_t = \tilde{\lambda}_n(\xi)$. So we reach the conclusion. For $i_1 \geqslant 1$, we consider the interval $(\tilde{\lambda}_t, \tilde{\lambda}_{t+i_1})$. If there are no eigenvalues of $U$ in this interval, we choose the cutting point $\xi$ just after $\tilde{\lambda}_{t+i_1}$, and in the same way we pair the eigenvalues and reach our conclusion. Assume that there are $i_2 - i_1 \geqslant 1$ eigenvalues of $U$ in this interval; then we consider the interval $(\tilde{\lambda}_{t+i_1}, \tilde{\lambda}_{t+i_2})$, and so on. We will obtain two conclusions by this procedure. Either we prove our theorem, or $(\tilde{\lambda}_{t+i_k}, \tilde{\lambda}_{t+i_{k+1}})$ includes $\lambda_1$ for some $i_k$. In the latter case, we can reduce the maximal difference $d$ (4.5) by pairing $\lambda_j$ with $\tilde{\lambda}_{t+j}$, which is a contradiction to our assumption. Figure 4 illustrates the proof. ■

## 5. CAUCHY INTERLACING THEOREM

It is well known that for Hermitian matrices the eigenvalues of a principal submatrix interlace those of the complete matrix. In more detail, if $A_k$ is a

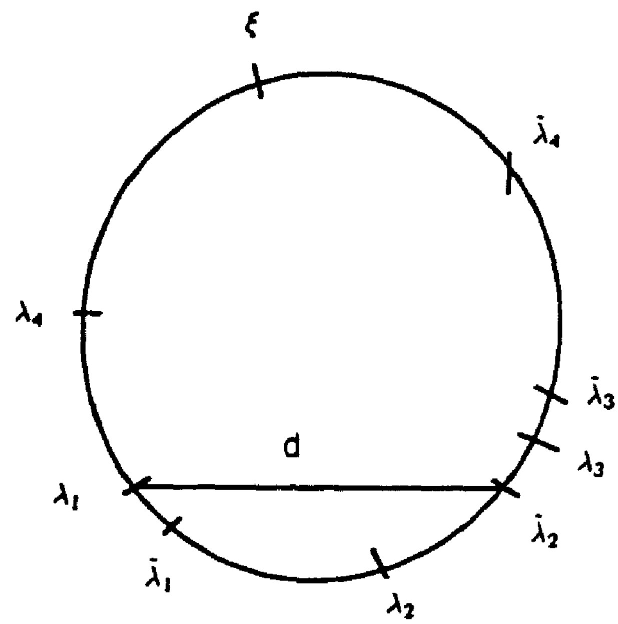


FIG. 4.

$k \times k$ principal submatrix of an $n \times n$ Hermitian matrix $A$, and

$$\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_k, \qquad \lambda_1 \leqslant \lambda_2 < \cdots \leqslant \lambda_n$$

are the eigenvalues of $A_k$ and $A$ respectively, then by Cauchy's interlacing theorem [23, 29]

$$\lambda_j \leqslant \mu_j \leqslant \lambda_{n+j-k}, \qquad j = 1, \dots, k.$$

For unitary matrices we cannot expect such a result, as a principal submatrix is not unitary any more. It is however possible to modify a $k \times k$ principal submatrix of an $n \times n$ unitary matrix $U$ so that the modified submatrix is unitary and its eigenvalues interlace those of $U$. To keep notation simple, we consider only leading principal submatrices. The case of general principal submatrices is similar.

DEFINITION. Let

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$$

be an $n \times n$ unitary matrix, and $U_{11}$ the $k \times k$ leading principal submatrix.

Then

$$U_k = U_{11} - U_{12}(I + U_{22})^+ U_{21} \tag{5.1}$$

is called the modified $k$th leading principal submatrix of $U$. Here $(I + U_{22})^+$ is the Moore-Penrose inverse of $I + U_{22}$.

Observe that when $-1$ is not an eigenvalue of $U_{22}$, $U_k$ is given by

$$U_k = U_{11} - U_{12}(I + U_{22})^{-1} U_{21}.$$

We have the following lemma.

LEMMA 5.1. *In the situation of the above definition, we have*

$$U_{12}(U_{22} + I)^+ (U_{22} + I) = U_{12}, \tag{5.2}$$

$$(U_{22} + I)(U_{22} + I)^+ U_{21} = U_{21}. \tag{5.3}$$

*Proof*. Obviously we need only to consider the case that $-1 \in \operatorname{Eig} U_{22}$. Let $x \neq 0$ be an $(n-k)$ vector such that $U_{22}x = -x$. Then, as $\begin{pmatrix} U_{12} \\ U_{22} \end{pmatrix}$ has orthogonal columns,

$$\|x\|^2 = \left\| \begin{pmatrix} U_{12}x \\ U_{22}x \end{pmatrix} \right\|^2 = \|x\|^2 + \|U_{12}x\|^2$$

and $U_{12}x = 0$. This shows that for the null spaces the inclusion $\operatorname{Ker}(U_{22} + I) \subset \operatorname{Ker} U_{12}$ holds. As $I - (U_{22} + I)^+(U_{22} + I)$ is the orthogonal projection onto $\operatorname{Ker}(U_{22} + I)$, (5.2) follows. Applying (5.2) to $U^H$ gives (5.3). ∎

THEOREM 5.2. *Let*

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$$

*be an $n \times n$ unitary matrix, $U_{11}$ the $k \times k$ leading principal submatrix of $U$, and*

$$U_k = U_{11} - U_{12}(I + U_{22})^+ U_{21}. \tag{5.4}$$

*Then*:

(i) $U_k$ *is unitary.*
(ii) *The matrix*

$$G = \begin{pmatrix} U_k^H & 0 \\ 0 & -I_{n-k} \end{pmatrix} U$$

*satisfies* $\operatorname{rank}(I - G) \leqslant n - k$.

(iii) *Let* $-1$ *be a eigenvalue of* $U_{22}$ *with multiplicity* $r$. *Then* $U_k$ *is the unique unitary matrix such that* $U = \operatorname{diag}(U_k, -I_{n-k})\, G$ *with* $\operatorname{rank}(I - G) = n - k - r$.

*Proof*. From $U^H U = I_n$ we have

$$\begin{aligned} U_{11}^H U_{11} + U_{21}^H U_{21} &= I_k, \\ U_{11}^H U_{12} + U_{21}^H U_{22} &= 0, \\ U_{12}^H U_{12} + U_{22}^H U_{22} &= I_{n-k}. \end{aligned} \tag{5.5}$$

Writing

$$\begin{aligned} U_k^H U_k = {} & U_{11}^H U_{11} - U_{21}^H \left(I + U_{22}^H\right)^+ U_{12}^H U_{11} \\ & - U_{11}^H U_{12} (I + U_{22})^+ U_{21} + U_{21}^H \left(I + U_{22}^H\right)^+ U_{12}^H U_{12} (I + U_{22})^+ U_{21} \end{aligned}$$

and replacing $U_{11}$ and $U_{12}$ by $U_{21}$ and $U_{22}$ according to (5.5) and also Lemma 5.1, we obtain

$$\begin{aligned} U_k^H U_k &= I - U_{21}^H U_{21} + U_{21}^H \left(I + U_{22}^H\right)^+ U_{22}^H U_{21} + U_{21}^H U_{22} (I + U_{22})^+ U_{21} \\ &\quad + U_{21}^H \left(I + U_{22}^H\right)^+ \left(I - U_{22}^H U_{22}\right)(I + U_{22})^+ U_{21} \\ &= I - U_{21}^H U_{21} + U_{21}^H \left(I + U_{22}^H\right)^+ \\ &\quad \times \left[U_{22}^H (I + U_{22}) + \left(I + U_{22}^H\right) U_{22} + \left(I - U_{22}^H U_{22}\right)\right] \\ &\quad \times (I + U_{22})^+ U_{21} \\ &= I - U_{21}^H U_{21} + U_{21}^H \left(I + U_{22}^H\right)^+ \left(I + U_{22}^H\right)(I + U_{22})(I + U_{22})^+ U_{21} \\ &= I. \end{aligned}$$

This shows (i). As by (5.2), (5.3)

$$U - \begin{pmatrix} U_k & 0 \\ 0 & -I_{n-k} \end{pmatrix}$$

$$= \begin{pmatrix} U_{12}(U_{22}+I)^{+} \\ I \end{pmatrix} (I + U_{22})\big((U_{22}+I)^{+} U_{21} \quad I\big),$$

we have that

$$\operatorname{rank}(G - I) = \operatorname{rank}\left[U - \begin{pmatrix} U_k & 0 \\ 0 & -I_{n-k} \end{pmatrix}\right] = n - k - r \leqslant n - k,$$

so we obtain (ii). The uniqueness of $U_k$ is easily seen from the proof. ■

In the case that $-1$ is not an eigenvalue of $U$ (and hence not a eigenvalue of $U_{22}$), we can explain the unitarity of $U_k$ in a more illuminating way. Observe that in this case $U_k + I$ is a Schur complement of $U + I$ with pivoting $U_{22} + I$, and hence $(U_k + I)^{-1}$ is the $k$th leading principal submatrix of $(U + I)^{-1}$ (e.g. Ouellette [28, (2.41)]). If $A = i(I + U)^{-1}(I - U)$ is the Cayley transformation of $U$, then equivalently

$$A + iI = 2i(I + U)^{-1}.$$

Taking the $k$th leading principal submatrices on both sides, we get

$$A_k + iI = 2i(I + U_k)^{-1},$$

where $A_k$ is the $k$th leading principal submatrix of $A$. This shows that $U_k$ is the inverse Cayley transformation of $A_k$ and hence unitary. Also, as the eigenvalues of $A_k$ interlace those of $A$, we have at once the interlacing result for the eigenvalues of $U_k$ and $U$.

THEOREM 5.3. *Suppose that $-1$ is not an eigenvalue of $U$ and that the angles corresponding to $\{\lambda_j\}_1^n$ and $\{\mu_j\}_1^k$, the eigenvalues of $U$ and $U_k$ respectively, are $\{\theta_j\}$ and $\{\tau_j\}$ satisfying*

$$-\frac{\pi}{2} < \theta_1 \leqslant \cdots \leqslant \theta_n < \frac{\pi}{2}$$

and

$$-\frac{\pi}{2} < \tau_1 \leqslant \tau_2 \leqslant \cdots \leqslant \tau_k < \frac{\pi}{2}$$

*respectively. Then*

$$\theta_j \leqslant \tau_j \leqslant \theta_{n+j-k}, \qquad j = 1, \dots, k.$$

*Proof.* The eigenvalues of $C(U) = A$ and $C(U_k) = A_k$ are given by

$$x_j = i\frac{1-\lambda_j}{1+\lambda_j} = \tan\theta_j, \qquad j = 1, \dots, n$$

$$\tilde{x}_j = i\frac{1-\mu_j}{1+\mu_j} = \tan\tau_j, \qquad j = 1, \dots, k.$$

As tan is monotone,

$$x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n, \qquad \tilde{x}_1 \leqslant \cdots \leqslant \tilde{x}_k.$$

and by Cauchy's interlacing theorem

$$x_j \leqslant \tilde{x}_j \leqslant x_{n+j-k}, \qquad j = 1, \dots, k.$$

By the monotonicity of arctan, we get

$$\theta_j \leqslant \tau_j \leqslant \theta_{n+j-k}. \qquad \blacksquare$$

Observe that this Cauchy interlacing theorem is also true when $-1 \in \operatorname{Eig} U$.

## 6. A PROPERTY OF BLOCK REFLECTORS

We conclude this paper by proving an interesting result on block reflectors. A unitary matrix $G$ such that $\operatorname{rank}(I - G) = k < n$ is called a block reflector. Such matrices have been studied in [8, 34]. Suppose that rank

$(I - G) = k$; then $G$ has the form

$$G = I - XDX^H, \tag{6.1}$$

where $X$ is an $n \times k$ matrix with $X^H X = I$, and $D$ is a diagonal matrix satisfying $D^H + D = D^H D$. This equation is equivalent to $D - I$ being unitary [8]. In [34], only the special case $\tilde{G} = I - 2XX^H$ is studied. There it is shown that the conditions

$$E^H E = F^H F \qquad (\textit{isometry property}), \tag{6.2}$$

$$E^H F = F^H E \qquad (\textit{symmetry property}) \tag{6.3}$$

are necessary and sufficient for the existence of a block reflector $\tilde{G} = I - 2XX^H$ such that $\tilde{G}E = F$. Here we show that the first property alone ensures that there is a block reflector $G$ with $\operatorname{rank}(I - G) \leqslant k$ such that $GE = F$.

THEOREM 6.1. *Suppose that $E$ and $F$ are two $n \times k$ $(k < n)$ matrices satisfying*

$$E^H E = F^H F. \tag{6.4}$$

*Then there exists a block reflector $G$ with* $\operatorname{rank}(I - G) \leqslant k$ *such that $E = G^H F$.*

*Proof.* First let us prove that Theorem 6.1 is true in the case that $F$ has the form

$$\begin{pmatrix} \tilde{F} \\ 0 \end{pmatrix},$$

where $\tilde{F}$ is a $k \times k$ square matrix. As $E^H E = F^H F$, there exists an unitary matrix $U$ such that $E = U^H(-F)$. Applying Theorem 5.2 to the unitary matrix $P^T U P$ for $n - k$, where

$$P = \begin{pmatrix} 0 & I_k \\ I_{n-k} & 0 \end{pmatrix},$$

we construct the modified $(n - k)$th leading principal submatrix $U_{n-k}$ as in

(5.1) and have the decomposition

$$P^T U P = \operatorname{diag}(U_{n-k}, -I_k)\,\tilde{G}$$

with $\operatorname{rank}(I - \tilde{G}) \leqslant k$, or equivalently

$$U = \operatorname{diag}(-I_k, U_{n-k})\,G.$$

Here $G = P\tilde{G}P^T$ and $\operatorname{rank}(I - G) \leqslant k$. So $G$ can be expressed as $G = I - XDX^H$. It follows immediately that $E = U^H(-F) = G^H F$. In the general case that $E^H E = F^H F$, let $F = QF_1$ be the $QR$ decomposition of $F$, where

$$F_1 = \begin{pmatrix} \tilde{F} \\ 0 \end{pmatrix},$$

and also $E_1 = Q^H E$. Then from $E_1^H E_1 = F_1^H F_1$ it follows that $E_1 = G^H F_1$. So $E = \tilde{G}^H F$, where $\tilde{G} = QGQ^H$ with $\operatorname{rank}(I - \tilde{G}) \leqslant k$. ∎

Observe that the block reflector $G$ can also be obtained directly by solving the equation $G^H F = E$.

## 7. CONCLUSIONS

In this paper, we have proved the following perturbation theorem: There exists a cutting point $\xi$ by which $\{\lambda_j(\xi)\}$ and $\{\tilde{\lambda}_j(\xi)\}$ are naturally ordered on the unit circle so that

$$\max_j \left|\lambda_j(\xi) - \tilde{\lambda}_j(\xi)\right| \leqslant \|U - \tilde{U}\|_2.$$

Basing on the Cayley transformation, we define a sequence of unitary submatrices of $U$, which are called the modified leading principal submatrices. Then we prove the Cauchy interlacing theorem: The eigenvalues of the modified submatrices interlace those of the complete matrix on the unit circle.

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