A Note on the Hoffman-Wielandt Theorem

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ABSTRACT

An elementary proof of a recent generalization of the Hoffman-Wielandt theorem to commuting \( m \)-tuples of normal matrices by Bhatia and Bhattacharyya is given. In the case \( m = 1 \) this provides a derivation, which, though using the same tools, avoids some unnecessary calculations and is thus shorter than the original one and that to be found in the standard literature.

Let \( A = (A^{(1)}, \ldots, A^{(m)}) \) be an \( m \)-tuple of commuting normal \( n \)-by-\( n \) matrices. There exists a unitary matrix \( U \) such that

\[
U^* A^{(j)} U = \Lambda_j = \text{diag}(\alpha_1^{(j)}, \ldots, \alpha_n^{(j)}), \quad j = 1, \ldots, m_F. \tag{1}
\]

The vector \( \alpha_k = (\alpha_k^{(1)}, \ldots, \alpha_k^{(m)}) \in \mathbb{C}^m, 1 \leq k \leq n, \) is called a joint eigenvalue of \( A \). In [1], Bhatia and Bhattacharyya prove:

**THEOREM.** Let \( A, B \) be two \( m \)-tuples of commuting normal operators with joint eigenvalues \( \alpha_k, \beta_k \), where \( \beta_k = (\beta_k^{(1)}, \ldots, \beta_k^{(m)}), 1 \leq k \leq n. \) There exist permutations \( \sigma, \tilde{\sigma} \) of \( \{1, \ldots, n\} \) such that

\[
\sum_{k=1}^n \| \alpha_k - \beta_{\sigma(k)} \|_2^2 = \sum_{k=1}^n \sum_{j=1}^m |\alpha_k^{(j)} - \beta_{\sigma(k)}^{(j)}|^2 \leq \sum_{j=1}^m \| A^{(j)} - B^{(j)} \|_F^2
\]

\[
\leq \sum_{k=1}^n \| \alpha_k - \beta_{\tilde{\sigma}(k)} \|_2^2 = \sum_{k=1}^n \sum_{j=1}^m |\alpha_k^{(j)} - \beta_{\tilde{\sigma}(k)}^{(j)}|^2. \tag{2}
\]

Here \( \| \cdot \|_2 \) is the Euclidean vector norm and \( \| \cdot \|_F \) the Frobenius matrix norm.

We give an elementary proof based only on

(i) the Birkhoff-König theorem stating that the set of doubly stochastic matrices is the convex hull of the permutation matrices,

(ii) the fact that a linear functional defined on a convex polyhedron achieves its optimal value on a set which includes a vertex.

**Proof.** Using (1) and

\[
V^H B^{(j)} V = M_j = \text{diag}(\beta_1^{(j)}, \ldots, \beta_n^{(j)}), \quad j = 1, \ldots, m, \tag{3}
\]

for a suitable unitary matrix \( V \), we get

\[
A^{(j)} - B^{(j)} = V(W \Lambda_j - M_j W) U^H,
\]

where \( W = V^H U = (w_{rs})_{r,s=1,\ldots,n} \) is unitary. As the Frobenius norm \( \| \cdot \|_F \) is unitarily invariant, we have

\[
\sum_{j=1}^m \| A^{(j)} - B^{(j)} \|_F^2 = \sum_{j=1}^m \| W \Lambda_j - M_j W \|_F^2 = \sum_{j=1}^m \sum_{r,s=1}^n |w_{rs}(\alpha_r^{(j)} - \beta_r^{(j)})|^2
\]

\[
= \sum_{r,s=1}^n |w_{rs}|^2 h_{rs}, \quad h_{rs} = \sum_{j=1}^m |\alpha_r^{(j)} - \beta_r^{(j)}|^2. \tag{4}
\]

Define the linear functional \( l(X) = \sum_{r,s=1}^n h_{rs} x_{rs} \) on the set \( \mathcal{M} \) of doubly stochastic matrices, and let \( \hat{W} = (|w_{rs}|^2) \in \mathcal{M} \). By (ii) above, \( l \) achieves its maximum and minimum on vertices of \( \mathcal{M} \), which by (i) are permutation matrices. Hence there are permutation matrices \( P_1, P_2 \) such that

\[
l(P_1) \leq l(\hat{W}) \leq l(P_2),
\]

i.e., there are permutations \( \sigma, \hat{\sigma} \) of \( \{1, \ldots, n\} \) such that

\[
\sum_{k=1}^n h_{\sigma(k),k} \leq l(\hat{W}) = \sum_{j=1}^m \| A^{(j)} - B^{(j)} \|_F^2 \leq \sum_{k=1}^n h_{\hat{\sigma}(k),k}. \tag{5}
\]

The equality here is shown in (4). But (5) is exactly the statement (2). \[\blacksquare\]
HOFFMAN-WIELANDT THEOREM

We remark that for $m = 1$ this is the famous Hoffman-Wielandt theorem; see [2] and many standard books on matrix theory. The proof given above is also slightly simpler than that in [2], where the convexity argument is applied to the expression $\| A - B \|^2_F - \| A \|^2_F - \| B \|^2_F$.

REFERENCES


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