A Note on the Variation of Permanents

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ABSTRACT

It is shown that for any two $n$-by-$n$ complex matrices $A, B$ the inequality

$$| \text{per}(A) - \text{per}(B) | \leq n\|A - B\| \max(\|A\|, \|B\|)^{n-1}$$

holds, if $\| \|$ is either the row-sum or the column-sum norm. It is conjectured that this result holds for any operator norm.

In [1], R. Bhatia proved that for any two $n$-by-$n$ matrices $A, B$ the inequality

$$| \text{per}(A) - \text{per}(B) | \leq n\|A - B\|_2 \max(\|A\|_2, \|B\|_2)^{n-1} \tag{1}$$

holds. Here $\| \|_2$ denotes the spectral norm.

The purpose of this note is to prove an analogous result for the row-sum and the column-sum norm. We recall that

$$\|A\|_\infty = \max_i \sum_k |a_{ik}|,$$

$$\|A\|_1 = \max_k \sum_i |a_{ik}|.$$
are the operator norms for the vector norms

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \quad \text{and} \quad \|x\|_\infty = \max_{i} |x_i|$$

respectively, where $A = (a_{ik})$, $x = (x_1, \ldots, x_n)^T$. We show

$$|\text{per}(A) - \text{per}(B)| \leq n \|A - B\|_p \max(\|A\|_p, \|B\|_p)^{n-1}, \quad p = 1, \infty. \quad (2)$$

As $\|A\|_1 = \|A^T\|_\infty$, it suffices to prove the case $p = 1$. We make use of the obvious inequality (see e.g. [3, p. 113])

$$|\text{per} A| \leq \|a_1\|_1 \|a_2\|_1 \cdots \|a_n\|_1,$$

where $A = (a_1, \ldots, a_n)$ and $a_i$ denotes the $i$th column of $A$. If $B = (b_1, \ldots, b_n)$, define

$$A_k = (a_1, a_2, \ldots, a_k, b_{k+1}, \ldots, b_n), \quad k = 1, \ldots, n-1,$$

$A_0 = B$, $A_n = A$. Then

$$|\text{per}(A_i) - \text{per}(A_{i-1})| = |\text{per}(a_1, \ldots, a_{i-1}, a_i - b_i, b_{i+1}, \ldots, b_n)|$$

$$\leq \|a_i - b_i\|_1 \prod_{j<i} \|a_j\|_1 \prod_{j>i} \|b_j\|_1.$$

Hence

$$|\text{per}(A) - \text{per}(B)| \leq \sum_{i=1}^{n} |\text{per}(A_i) - \text{per}(A_{i-1})|$$

$$\leq n \max_{i} \|a_i - b_i\|_1 \max_{i} \left( \prod_{j<i} \|a_j\|_1 \prod_{j>i} \|b_j\|_1 \right)$$

$$\leq n \|A - B\|_1 \max(\|A\|_1, \|B\|_1)^{n-1}.$$

This establishes (2).

We remark that this proof is completely elementary, while that in [1] uses nontrivial tools from multilinear algebra.
Also, (1) and (2) are not comparable, as different operator norms are not comparable.

It is tempting to state the

CONJECTURE. If \(|\|\|\) denotes any operator norm for \(n\)-by-\(n\) matrices, then

\[
\|\text{per}(A) - \text{per}(B)\| \leq n\|A - B\|\max(\|A\|, \|B\|)^{n-1}.
\]

(3)

We remark finally that S. Friedland has proved (3) for the determinant function instead of the permanent [2].

Note added in proof. S. Friedland has shown (private communication) the following related result, which implies (1) and is near to the conjecture: For any operator norm \(\|\|\|

\[
\|\text{per}(A) - \text{per}(B)\| \leq \frac{n}{2} \left[ \|A - B\|\max(\|A\|, \|B\|)^{n-1} \\
+ \|A^* - B^*\|\max(\|A^*\|, \|B^*\|)^{n-1}\right].
\]

REFERENCES


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