Combinatorics, Probability and Computing (1993) 2, 211–220

On Extremal Set Partitions in Cartesian Product Spaces

RUDOLF AHLSWEDE and NING CAI

Universität Bielefeld, Fakultät für Mathematik, Postfach 100131, 33501 Bielefeld, Germany

Received 8 April 1993; revised 18 August 1993

For Paul Erdős on his 80th birthday

The partition number of a product hypergraph is introduced as the minimal size of a partition of its vertex set into sets that are edges. This number is shown to be multiplicative if all factors are graphs with all loops included.

1. Introduction

Consider $(\mathcal{V},\mathcal{E})$, where $\mathcal{V}$ is a finite set and $\mathcal{E}$ is a system of subsets of $\mathcal{V}$. For the cartesian products $\mathcal{V}^n = \prod_1^n \mathcal{V}$ and $\mathcal{E}^n = \prod_1^n \mathcal{E}$, let $\pi(n)$ denote the minimal size of a partition of $\mathcal{V}^n$ into sets that are elements of $\mathcal{E}^n$ if a partition exists at all, otherwise $\pi(n)$ is not defined. This is obviously exactly the case if it is so for $n = 1$.

Whereas the packing number $p(n)$, that is the maximal size of a system of disjoint sets from $\mathcal{E}^n$, and the covering number $c(n)$, that is the minimal number of sets from $\mathcal{E}^n$ to cover $\mathcal{V}^n$, have been studied in the literature, this seems to be not the case for the partition number $\pi(n)$.

Obviously, $c(n) \leq \pi(n) \leq p(n)$, if $c(n)$ and $\pi(n)$ are well defined. The quantity $\lim_{n\to\infty} \frac{1}{n} \log p(n)$ is Shannon's zero error capacity [11]. Although it is known only for very few cases (see [7]), a nice formula exists for $\lim_{n\to\infty}(1/n)\log c(n)$ (see [1, 10]).

The difficulties in analyzing $\pi(n)$ are similar to those for $p(n)$. For the case of graphs with edge set $\mathcal{E}$ including all loops, we prove that $\pi(n) = \pi(1)^n$ (Theorem 3). This result is derived from the corresponding result for complete graphs (Theorem 2) with the help of Gallai's Lemma in matching theory [6]. More general results concern products of hypergraphs with non-identical factors. Another interesting quantity is $\mu(n)$, the maximal size of a partition of $\mathcal{V}^n$ into sets that are elements of $\mathcal{E}^n$ (again only hypergraphs $(\mathcal{V},\mathcal{E})$ with a partition are considered). We also call μ the maximal partition number. It behaves more like the packing number (see example 5). Clearly, $\pi(n) \leq \mu(n) \leq p(n)$. It seems to us that an understanding of these partition problems would be a significant contribution to an understanding of the basic, and seemingly simple, notion of Cartesian products. Another partition problem was formulated in [12]. Among the contributions to this problem, we refer the reader to [5], [9], and [12].

2. Products of complete graphs: first results

For a complete graph $\mathscr{C} = \{\mathscr{V}, \mathscr{E}\}$, let $\mathscr{E}^* = \mathscr{E} \cup \{\{v\} : v \in \mathscr{V}\}$ and define the hypergraph $\mathscr{C}^n = \{\mathscr{V}^n, \mathscr{E}^n\}$, where $\mathscr{V}^n = \prod_1^n \mathscr{V}$ and $\mathscr{E}^n = \prod_1^n \mathscr{E}^*$.

We study the partition number $\pi(n)$, first for $\mathscr{C}^n$, and in later sections extend our results to hypergraphs, which are products of arbitrary graphs including all loops.

First we introduce the map $\sigma : \mathscr{E}^n \to \{0,1\}^n$, where

$$s^n = \sigma(E^n) = (\log|E_1|, \dots, \log|E_n|). \tag{2.1}$$

As weight of E^n ($w(E^n)$ for short), we choose the Hamming weight $w_H(s^n) = \sum_{t=1}^n s_t$. Notice that the cardinality $|E^n|$ equals $2^{w(E^n)}$.

Instead of partitions, we consider more generally a packing $\mathscr{P}$ of $\mathscr{C}^n$. We set

$$\mathscr{P}_i = \{E^n \in \mathscr{P} : w(E^n) = i\}, P_i = |\mathscr{P}_i|, \tag{2.2}$$

and call $\{P_i\}_{i=0}^n$ the weight distribution of $\mathscr{P}$.

We associate with $\mathscr{P}$ the set of shadows $\mathscr{Q} \subset \mathscr{X}^n$ defined by

$$\mathscr{Q} = \{E^n \in \mathscr{E}^n : E^n \subset F^n \text{ for some } F^n \in \mathscr{P}\}, \tag{2.3}$$

and its level sets

$$\mathscr{Q}_i = \{E^n \in \mathscr{Q} : w(E^n) = i\}, 0 \le i \le n. \tag{2.4}$$

It is convenient to write $Q_i = |\mathscr{Q}_i|$. $\{Q_i\}_{i=0}^n$ is the weight distribution of $\mathscr{Q} = \text{shad}(\mathscr{P})$.

First we establish some simple connections between these weight distributions.

Lemma 1. *For a packing $\mathscr{P}$ of $\mathscr{C}^n$*

$$\sum_{i=k}^n 2^{i-k} \binom{i}{k} P_i = Q_k. \tag{2.5}$$

Proof. Consider any edge E^n with weight $w(E^n) = i \ge k$. There are exactly $2^{i-k}\binom{i}{k}$ edges contained in E^n with weight k. Therefore we have always

$$\sum_{i=k}^n 2^{i-k} \binom{i}{k} P_i \ge Q_k. \tag{2.6}$$

□

Lemma 2. *For a packing $\mathscr{P}$ of $\mathscr{C}^n$*

$$|\mathscr{P}| = \sum_{i=0}^n P_i = \sum_{k=0}^n (-1)^k Q_k. \tag{2.7}$$

Proof. An edge $E^n \in \mathcal{P}_i$ contributes to $\sum_{k=0}^{n}(-1)^k Q_k$ the amount

$$\sum_{k=0}^{i}(-1)^k 2^{1-k}\binom{i}{k} = (2-1)^i = 1.$$

□

Lemma 3. *For a packing $\mathcal{P}$ of $\mathcal{C}^n$*

$$P_0 = \sum_{k=0}^{n}(-1)^k 2^k Q_k \tag{2.8}$$

and if in addition $\mathcal{P}$ is a partition and $S = |\mathcal{V}|$ is odd,

$$\sum_{k=0}^{n}(-1)^k 2^k Q_k - 1 \geq 0. \tag{2.9}$$

Proof. An edge $E^n \in \mathcal{P}_i$ contributes to $\sum_{k=0}^{n}(-1)^k 2^k Q_k$ the amount

$$\sum_{k=0}^{i}(-1)^k 2^k 2^{i-k}\binom{i}{k} = 2^i(1-1)^i,$$

which equals 1, if $i = 0$, and 0, otherwise.

Therefore (2.8) holds.

Furthermore, if S is odd, then so is S^n and there must be an edge in the partition of odd size, that is, $P_0 \geq 1$ or, equivalently, by (2.8), (2.9) must hold. □

Remark 1. The last two Lemmas can be derived more systematically from Lemma 1 by Möbius Inversion. Here this machinery can be avoided, but we need it for the more abstract setting of [4].

3. Products of complete graphs: the main results

We shall now exploit Lemma 3 by applying it to classes of subhypergraphs, which we now define. For any $I \subset \{1, 2, \ldots, n\}$ and any specification $(v_j)_{j \in I^c}$, where $v_j \in \mathcal{V}_j$, we set

$$\mathcal{C}^n\big(I, (v_j)_{j \in I^c}\big) = \left(\prod_{i=1}^{n}\mathcal{U}_i, \prod_{i=1}^{n}\mathcal{F}_i\right) = \big(\mathcal{U}^n, \mathcal{F}^n\big), \tag{3.1}$$

where

$$\mathcal{U}_i = \begin{cases} \mathcal{V}_i \\ \{v_i\} \end{cases} \text{ and } \mathcal{F}_i = \begin{cases} \mathcal{E}_i & \text{for } i \in I \\ \{v_i\} & \text{for } i \in I^c. \end{cases} \tag{3.2}$$

Clearly, for a partition $\mathcal{P}$ of $\mathcal{C}^n$ and $\mathcal{Q} = \text{shad}\mathcal{P}$, the set $\mathcal{Q}\big(I, (v_j)_{j \in I^c}\big) = \mathcal{Q} \cap \mathcal{F}^n$ is a downset, and the map

$$\psi : \mathcal{F}^n \to \prod_{i \in I}\mathcal{E}_i,\ \psi\left(\prod_{i=1}^{n}E_i\right) = \prod_{i \in I}E_i \tag{3.3}$$

is a bijection.

Write $\tilde{\mathscr{Q}} = \psi(\mathscr{Q} \cap \mathscr{F}^n)$ and let $\tilde{\mathscr{Q}}_i$ count the members of $\tilde{\mathscr{Q}}$ of weight i. Since $\tilde{\mathscr{Q}}$ is a downset in $\prod_{i \in I} \mathscr{E}_i$ and its maximal elements form a partition of $\prod_{i \in I} \mathscr{V}_i$, we know that $\tilde{\mathscr{Q}}_0 = S^m$. This fact and Lemma 3 yield

$$S^m + \sum_{k=1}^{m} (-1)^k 2^k \tilde{\mathscr{Q}}_k - 1 \geq 0. \tag{3.4}$$

This is the key to the proof of the following important result.

Theorem 1. *For a partition $\mathscr{P}$ of $\mathscr{C}^n = (\mathscr{V}^n, \mathscr{E}^n)$ with $\mathscr{V}^n = \prod_{i=1}^n \mathscr{V}_i$, $|\mathscr{V}_i| = S$ for $i = 1, 2, \ldots, n$, the weight distribution $(Q_k)_{k=0}^n$ of $Q = shad\mathscr{P}$ satisfies, for $1 \leq m \leq n$,*

$$\binom{n}{m} S^n + \sum_{k=1}^{m} (-1)^k \binom{n-k}{m-k} 2^k Q_k - \binom{n}{m} S^{n-m} \geq 0. \tag{3.5}$$

Proof. The map ψ preserves inclusions and weights. The total number of pairs $(I, (v_j)_{j \in I^c})$ with $|I| = m$ equals $\binom{n}{m} S^{n-m}$. Moreover, each $E^n \in \mathscr{Q}$ with $w(E^n) = k$ is contained in exactly $\binom{n-k}{m-k}$ sets of the form $\mathscr{Q}(I, (v_j)_{j \in I^c})$ and thus for the sets of weight k

$$\binom{n-k}{m-k} Q_k = \sum_{(I, (v_j)_{j \in I^c}), |I| = m} \left| \mathscr{Q}_k(I, (v_j)_{j \in I^c}) \right| . \tag{3.6}$$

We have one equation of the form (3.4) for each pair $(I, (v_j)_{j \in I_c})$. Summation of their left-hand sides gives, therefore,

$$\binom{n}{m} S^{n-m} \cdot S^m + \sum_{k=1}^{m} (-1)^k 2^k \binom{n-k}{m-k} Q_k - \binom{n}{m} S^{n-m} \geq 0$$

and hence (3.5). □

Now comes the harvest.

Theorem 2. *For a partition $\mathscr{P}$ of $\mathscr{C}^n$*

$$|\mathscr{P}| \geq \left\lceil \frac{S}{2} \right\rceil^n .$$

Proof. Since $|E^n| \leq 2^n$, obviously $|\mathscr{P}| \geq S^n / 2^n$, and for $S = 2\alpha$ even, the result obviously holds. Now let $S = 2\alpha + 1$.

Summing the left-hand side expressions in (3.5) for $m = 1, 2, \ldots, n$ results in

$$\sum_{m=1}^{n} \binom{n}{m} S^n + \sum_{m=1}^{n} \sum_{k=1}^{m} (-1)^k \binom{n-k}{m-k} 2^k Q_k - \sum_{m=1}^{n} \binom{n}{m} S^{n-m} \geq 0,$$

or in

$$(2^n-1)S^n+\sum_{k=1}^{n}(-1)^k 2^k Q_k \sum_{m=k}^{n}\binom{n-k}{m-k}-\left[(S+1)^n-S^n\right]\geq 0.$$

This is equivalent to

$$2^n\cdot\left[S^n+\sum_{k=1}^{n}(-1)^k Q_k\right]-(S+1)^n\geq 0.$$

As $Q_0=S^n$, we conclude, with Lemma 2,

$$P\geq (S+1)^n\cdot 2^{-n}=\left\lceil\frac{S}{2}\right\rceil^n,\quad \text{if } S \text{ is odd.}$$

□

4. Non-identical factors: a generalization

We now consider hypergraphs $\mathscr{C}'^n$ with vertex sets $\mathscr{V}^n=\prod_{t=1}^{n}\mathscr{V}_t$ and edge sets $\mathscr{E}^n=\prod_{t=1}^{n}\mathscr{E}_t$, where the $\mathscr{V}_t$'s are finite sets of not necessarily equal cardinalities S_t. The factors $\mathscr{E}_t$ are such that $(\mathscr{V}_t,\mathscr{E}_t)$ is a complete graph with all loops included. We shall write, with positive integers α_t,

$$|\mathscr{V}_t|=2\alpha_t+\varepsilon_t,\ \varepsilon_t\in\{0,1\}. \tag{4.1}$$

Inspection shows that the sizes of factors do not affect the proofs of Lemmas 1 and 2. Also (2.8) in Lemma 2 holds and since $P_0\geq 1$, if $\varepsilon_t=1$ for $t=1,2,\ldots,n$, we can generalize (2.9) to

$$\sum_{k=0}^{n}(-1)^k 2^k Q_k-\prod_{k=1}^{n}\varepsilon_k\geq 0. \tag{4.2}$$

Theorem 1 in Section 3 generalizes to

Theorem 1′. *For a partition $\mathscr{P}$ of $\mathscr{C}'^n$*

$$\binom{n}{m}\prod_{i=1}^{n}S_i+\sum_{k=1}^{m}(-1)^k\binom{n-k}{m-k}2^k Q_k-\sum_{I:|I|=m}\prod_{i\in I}\varepsilon_i\prod_{j\in I^c}S_j\geq 0. \tag{4.3}$$

Proof. **(Sketch)** In the proof of Theorem 1, replace S^m by $\prod_{i\in I}S_i$ and inequality (3.4) by

$$\prod_{i\in I}S_i+\sum_{k=1}^{n}(-1)^k 2^k\tilde{Q}_k-\prod_{i\in I}\varepsilon_i\geq 0. \tag{4.4}$$

□

Theorem 2′. *For a partition $\mathscr{P}$ of $\mathscr{C}'^n$*

$$|\mathscr{P}|\geq\prod_{i=1}^{n}\left\lceil\frac{S_i}{2}\right\rceil. \tag{4.5}$$

Proof. Summing the expressions on the left-hand side in (4.3) for $m = 1,2,\dots,n$ results in

$$\begin{aligned} 0 &\le \sum_{m=1}^{n}\binom{n}{m}\prod_{i=1}^{n} S_i + \sum_{m=1}^{n}\sum_{k=1}^{m}\binom{n-k}{m-k}(-1)^k 2^k Q_k - \sum_{m=1}^{n}\sum_{I:|I|=m}\prod_{i\in I}\varepsilon_i\prod_{j\in I^c} S_j \\ &= (2^n-1)\prod_{i=1}^{n} S_i + \sum_{k=1}^{n}(-1)^k 2^k Q_k \sum_{m=k}^{n}\binom{n-k}{m-k} - \sum_{\phi\neq I}\prod_{i\in I}\varepsilon_i\prod_{j\in I^c} S_i \\ &= 2^n\left[\prod_{i=1}^{n} S_i + \sum_{k=1}^{n}(-1)^k Q_k\right] - \sum_{I}\prod_{i\in I}\varepsilon_i\prod_{j\in I^c} S_j \end{aligned}$$

or

$$|\mathscr{P}| \ge 2^{-n}\sum_{I}\prod_{i\in I}\varepsilon_i\prod_{j\in I^c} S_j. \tag{4.6}$$

We evaluate the expression on the right-hand side by introducing $J = \{\ell : 1 \le \ell \le n, \varepsilon_\ell = 1\}$ and $I^* = J \setminus I$. Then

$$\sum_{I}\prod_{i\in I}\varepsilon_i\prod_{j\in I^c} S_j = \sum_{I\subset J}\prod_{j\in I^*} S_j\cdot\prod_{j\in J^c} S_j$$

$$= \prod_{j\in J}(S_j+1)\cdot\prod_{j\in J^c} S_j = \prod_{j=1}^{n}(S_j+\varepsilon_j) \text{ and (4.5) follows.}$$ □

Corollary 1. *The partition number $\pi(\mathscr{C}'^n)$ equals $\prod_{j=1}^{n}\left\lceil\frac{S_j}{2}\right\rceil$.*

Proof. The partition number of $(\mathscr{V}_j,\mathscr{E}_j)$ is $\left\lceil\frac{S_j}{2}\right\rceil$. Take a product of optimal partitions for the factors. This construction gives the lower bound in Theorem 2′. □

5. Products of general graphs

We assume now that the factors $\mathscr{G}_t = (\mathscr{V}_t,\mathscr{E}_t)$ $(t = 1,2,\dots,n)$ are arbitrary finite graphs with all loops included.

Obviously, we have for the partition number

$$\pi(\mathscr{G}_t) = |\mathscr{V}_t| - \nu(\mathscr{G}_t), \tag{5.1}$$

where $\nu(\mathscr{G}_t)$ is the matching number of $\mathscr{G}_t$.

Theorem 3. *For the hypergraph product $\mathscr{H}^n = \mathscr{G}_1\times\dots\times\mathscr{G}_n$*

$$\pi(\mathscr{H}^n) = \prod_{t=1}^{n}\pi(\mathscr{G}_t). \tag{5.2}$$

Here only the inequality

$$\pi(\mathscr{H}^n) \geq \prod_{t=1}^{n} \pi(\mathscr{G}_t) \tag{5.3}$$

is non-trivial. We make use of a well-known result from matching theory.

Gallai's Lemma. *([6] or [8] page 89) If a graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ is connected and, for all $v \in \mathscr{V}$, $\nu(\mathscr{G} - v) = \nu(\mathscr{G})$, then $\mathscr{G}$ is factor-critical, that is, for all $v \in \mathscr{V}$, $\mathscr{G} - v$ has a perfect matching.*

Proof of 5.3. For every $t \in \{1, 2, \ldots, n\}$ we modify $\mathscr{G}_t$ as follows: remove any vertex $v \in \mathscr{V}_t$ with $\nu(\mathscr{G}_t - v) < \nu(\mathscr{G}_t)$ and reiterate this until a graph $\mathscr{G}_t^*$ with $\nu(\mathscr{G}_t^* - v) = \nu(\mathscr{G}_t^*)$ for all $v \in \mathscr{V}_t^*$ is obtained.

Notice that (5.1) ensures that

$$\pi(\mathscr{G}_t) = \pi(\mathscr{G}_t^*). \tag{5.4}$$

Denote the set of connected components of $\mathscr{G}_t^*$ by $\{\mathscr{G}_t^{*(j)}\}_{j \in J_t}$. Clearly,

$$\pi(\mathscr{G}_t^*) = \sum_{j \in J_t} \pi\left(\mathscr{G}_t^{*(j)}\right). \tag{5.5}$$

Moreover, by Gallai's Lemma each component $\mathscr{G}_t^{*(j)}$ has a vertex set $\mathscr{V}_t^{*(j)}$ of odd size and

$$\nu\left(\mathscr{G}_t^{*(j)}\right) = \left(|\mathscr{V}_t^{*(j)}| - 1\right)2^{-1} \triangleq \alpha_t^j, \text{ say.}$$

Thus,

$$\pi(\mathscr{G}_t^*) = \sum_j (\alpha_t^j + 1). \tag{5.6}$$

Now, for $\mathscr{H}^{*n} = \prod_1^n \mathscr{G}_t^*$ we have

$$\pi(\mathscr{H}^n) \geq \pi(\mathscr{H}^{*n}), \tag{5.7}$$

because the modifications described above transform a partition of $\mathscr{H}^n$ into a partition of $\mathscr{H}^{*n}$ with no more parts.

Finally, by Theorem 2′, we have for the product $\mathscr{C}^n$ of complete graphs with vertex sets $\mathscr{V}_t^{*(j)}$ that

$$\pi\left(\mathscr{G}_1^{*(j_1)} \times \ldots \times \mathscr{G}_n^{*(j_n)}\right) \geq \pi(\mathscr{C}^n) = (\alpha_1^{j_1} + 1) \ldots (\alpha_n^{j_n} + 1). \tag{5.8}$$

Therefore,

$$\begin{aligned} \pi(\mathscr{H}^{*n}) &= \sum_{j_1 \in J_1, \ldots, j_n \in J_n} \pi\left(\mathscr{G}_1^{*(j_1)} \times \ldots \times \mathscr{G}_n^{*(j_n)}\right) \\ &\geq \sum_{(j_1, \ldots, j_n)} (\alpha_1^{j_1} + 1) \ldots (\alpha_n^{j_n} + 1) \end{aligned}$$

$$= \prod_{t=1}^{n} \sum_{j \in J_t} (\alpha_t^j + 1)$$

$$= \prod_{t=1}^{n} \pi(\mathscr{G}_t^*) = \prod_{t=1}^{n} \pi(\mathscr{G}_t).$$

This and (5.7) imply (5.3). □

6. Examples for deviation from multiplicative behaviour

First we give two examples of product hypergraphs $\mathscr{H} \times \mathscr{H}'$ for which the partition number π is not multiplicative in the factors. They are due to K.-U. Koschnick.

Example 1.

$$\mathscr{V}_1 = \{0,1,2,\dots,6\}, \mathscr{E}_1 = \{E \subseteq \mathscr{V}_1 : |E| \in \{1,4\}\}.$$

Clearly, $\pi(\mathscr{H}_1) = 4$ and the partition

$$\begin{aligned}\{\{i\} \times \{0,1,2,3\} : i = 0,1,2\} \;&\cup\; \{\{i\} \times \{3,4,5,6\} : i = 4,5,6\} \\ &\cup\; \{\{0,1,2,3\} \times \{j\} : j = 4,5,6\} \\ &\cup\; \{\{3,4,5,6\} \times \{j\} : j = \{0,1,2\}\} \\ &\cup\; \{\{3\} \times \{3\}\}\end{aligned}$$

has 13 members. Therefore

$$\pi(\mathscr{H}_1 \times \mathscr{H}_1) \le 13 < \pi(\mathscr{H}_1)\pi(\mathscr{H}_1) = 16. \tag{6.1}$$

While this example seems to be the smallest possible for identical factors, one can do better with non-identical factors:

$$\mathscr{H}_1 \times \mathscr{H}_1', \text{ where } \mathscr{V}_1' = \{0,1,2,3,4\} \text{ and } \mathscr{E}_1' = \{E \subset \mathscr{V}_1' : |E| \in \{1,3\}\}.$$

Here, by a similar construction, $\pi(\mathscr{H}_1 \times \mathscr{H}_1') \le 11$, whereas $\pi(\mathscr{H}_1) \cdot \pi(\mathscr{H}_1') = 4 \cdot 3 = 12$.

Example 2. Since π is multiplicative for graphs, one may wonder whether it is multiplicative if one factor is a graph.

Consider $G = (\mathscr{V}, \mathscr{E})$ with $\mathscr{V} = \{0,1,\dots,4\}$ and $\mathscr{E} = \{\{i, i+1 \bmod 5\} : i = 0,1,\dots,4\} \cup \{i : 0 \le i \le 4\}$, that is, the pentagon with all loops.

Define $\mathscr{H}' = (\mathscr{V}', \mathscr{E}')$ with $\mathscr{V}' = \{1,2,\dots,14\}$ and $\mathscr{E}' = \{E \subset \mathscr{V}' : |E| \in \{1,9\}\}$.

Notice that $\pi(G) = 3$, $\pi(\mathscr{H}') = 7$, and that the following construction ensures $\pi(G \times \mathscr{H}') \le 20 < 21 = \pi(G) \cdot \pi(\mathscr{H}')$:

$$\begin{aligned}&\{\{i\} \times \{j + k \bmod 14 : 0 \le k \le 8\} : (i,j) \in \{(0,0),(1,3),(2,6),(3,9),(4,12)\}\} \\ &\cup\; \{\{1,2\} \times \{j\} : j = 0,1,2\} \cup \{\{2,3\} \times \{j\} : j = 2,3,5\} \\ &\cup\; \{\{3,4\} \times \{j\} : j = 6,7,8\} \\ &\cup\; \{\{4,0\} \times \{j\} : j = 9,10,11\} \\ &\cup\; \{\{0,1\} \times \{j\} : j = 12,13,14\}\end{aligned}$$

is a set of $5 + 5 \cdot 3 = 20$ edges partitioning $\mathscr{V} \times \mathscr{V}'$.

To help orient the reader, we add three examples, which demonstrate that the covering number c, the packing number p and the maximal partition number μ are not multiplicative in the factors either.

Example 3. $\mathscr{V}_3 = \{0,1,2\}$, $\mathscr{E}_3 = \{E \subseteq \mathscr{V} : |E| = 2\}$

We have

$$3 = c(\mathscr{H}_3 \times \mathscr{H}_3) \neq c(\mathscr{H}_3) \cdot c(\mathscr{H}_3) = 4, \tag{6.2}$$

because $\mathscr{C}\{\{0,1\} \times \{0,1\}, \{0,2\} \times \{0,2\}, \{1,2\} \times \{1,2\}\}$ covers $\mathscr{V}_3 \times \mathscr{V}_3$ and there is no covering with 2 edges.

This is the smallest example in terms of the number of vertices.

Remark 2. Quite generally, even in the case of non-identical factors $\mathscr{H}_t = (\mathscr{V}_t, \mathscr{E}_t)$, $t \in \mathbb{N}$, with $\max_t |\mathscr{E}_t| < \infty$, the asymptotic behaviour of $c(n)$ is known [1]:

$$\lim_{n\to\infty} \frac{1}{n} \left(\log c(n) - \sum_{t=1}^{n} \log \left(\max_{q \in \mathrm{Prob}(\mathscr{E}_t)} \min_{v \in \mathscr{E}_t} \sum_{E \in \mathscr{E}_t} 1_E(v) q_E \right)^{-1} \right) = 0,$$

where $\mathrm{Prob}(\mathscr{E}_t)$ is the set of all probability distributions on $\mathscr{E}$, q_E is the probability of E under q and 1_E is the indicator function of the set E.

Example 4. $\mathscr{V}_4 = \{0,1,2,3,4\}, \mathscr{E}_4 = \big\{\{x, x+1 \bmod 5\} : x \in \mathscr{V}_4\big\}$.

Here we have

$$5 = p(\mathscr{H}_4 \times \mathscr{H}_4) \neq p(\mathscr{H}_4) p(\mathscr{H}_4) = 4. \tag{6.3}$$

It was shown in [11] that this is the smallest example in the previous sense. Notice that it is bigger than the previous one.

Example 5. To avoid heavy notation, we will write $\mathscr{H}_5 = (\mathscr{V}_5, \mathscr{E}_5)$ without an index as $\mathscr{H} = (\mathscr{V}, \mathscr{E})$. It is made up of the 5 vertex sets

$$\mathscr{W}_i = \{x_{ij} : j = 1,2,\ldots,m\}, 3 \leq m (i = 0,1,2,\ldots,4),$$

the 6 edge sets

$$\mathscr{G}_i = \big\{(x_{ij}, x_{i+1 \bmod 5, j}) : j = 1,2,\ldots,m\big\} (i = 0,1,2,\ldots,4),$$

and $\{\mathscr{W}_0, \ldots, \mathscr{W}_4\}$. Thus

$$\mathscr{V} = \bigcup_{i=0}^{4} \mathscr{W}_i, \mathscr{E} = \{\mathscr{W}_0, \ldots, \mathscr{W}_4\} \cup \Big(\bigcup_{i=0}^{4} \mathscr{G}_i\Big).$$

A look at the pentagon with vertex set $\{x_{01}, x_{11}, x_{21}, x_{31}, x_{41}\}$ shows that a partition of $\mathscr{H}$ must contain at least one of the edges $\mathscr{W}_i$ as a member. On the other hand, the vertices $\mathscr{V} \setminus \mathscr{W}_i$ have a maximal partition of size $2m$. Therefore we have shown that $\mu(\mathscr{H}) = 2m + 1$. We shall next consider $\mu(\mathscr{H} \times \mathscr{H})$. For this we introduce the superedges

$$\mathscr{G}_i^* = \mathscr{W}_i \cup \mathscr{W}_{i+1 \bmod 5} (i = 0,1,\ldots,4)$$

in $\mathscr{H}$, and the superedges $\mathscr{G}_i^* \times \mathscr{G}_{i'}^* (i, i' = 0,1,\ldots,4)$ in $\mathscr{H} \times \mathscr{H}$. Whereas $\mathscr{G}_i^*$ can be partitioned into m edges, they can be partitioned into m^2 edges.

First we divide $\mathcal{V} \times \mathcal{V}$ into 25 parts $\{\mathcal{W}_i \times \mathcal{W}_{i'} : i, i' = 0, 1, \ldots, 4\}$. Then we pack 5 superedges (as in Shannon's construction) into $\mathcal{V} \times \mathcal{V}$. They cover 20 parts, and the remaining 5 parts are packed with 5 edges of type $\mathcal{W}_i \times \mathcal{W}_{i'}$. Finally, we partition the 5 superedges into the edges of $\mathcal{H} \times \mathcal{H}$. Thus we obtain a desired partition with $5 + 5m^2$ edges. Notice that $\mu(\mathcal{H} \times \mathcal{H}) \geq 5 + 5m^2 > (2m+1)^2 = \mu(\mathcal{H})^2$ for $m \geq 3$. The smallest example in this class has 15 vertices.

Remark 3. The construction was based on the pentagon. Its vertices were replaced by sets of vertices $\mathcal{W}_i$ with a numbering. The vertices with the same number in the $\mathcal{W}_i$'s form a pentagon. Thus we obtained $m = |\mathcal{W}_i|$ many pentagons. Then we added the $\mathcal{W}_i'$ as further edges. Finally we used the superedges to mimic the original small edges. We can make this construction starting with any hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$. If it has the property $p(\mathcal{H})^2 < p(\mathcal{H} \times \mathcal{H})$, then for m large enough our construction gives an associated hypergraph for which μ is not multiplicative.

7. Acknowledgement

The authors are very much indebted to Klaus-Uwe Koschnick for constructing beautiful examples.

References

[1] Ahlswede, R. (preprint) On set coverings in Cartesian product spaces, Manuscript 1971. Reprinted in *SFB 343 Diskrete Strukturen in der Mathematik*, Preprint 92–034.

[2] Ahlswede, R. (1979) Coloring hypergraphs: A new approach to multi-user source coding, Pt I. *Journ. of Combinatorics, Information and System Sciences* **4** (1) 76–115.

[3] Ahlswede, R. (1980) Coloring hypergraphs: A new approach to multi-user source coding, Pt II. *Journ. of Combinatorics, Information and System Sciences* **5** (3) 220–268.

[4] Ahlswede, R. and Cai, N. (preprint) On POS partition and hypergraph products. *SFB 343 Diskrete Strukturen in der Mathematik*, Preprint 93–008.

[5] Ahlswede, R., Cai, N. and Zhang, Z. (1989) A general 4-words inequality with consequences for 2-way communication complexity. *Advances in Applied Mathematics* **10** 75–94.

[6] Gallai, T. (1963) Neuer Beweis eines Tutte'schen Satzes. *Magyar Tud. Akad. Mat. Kutato Int. Közl.* **8** 135–139.

[7] Lovasz, L. (1979) On the Shannon capacity of a graph. *IEEE Trans. Inform. Theory* **IT-25** 1–7.

[8] Lovász, L. and Plummer, M. D. (1986) *Matching Theory*, North-Holland Mathematics studies 121, North-Holland.

[9] Mehlhorn, K. and Schmidt, E. M. (1982) Las Vegas is better than determinism in VLSI and distributed computing. *Proceedings 14th ACM STOC* 330–337.

[10] Posner, E. C. and Mc Eliece, R. J. (1971) Hide and seek, data storage and entropy. *Annals of Math. Statistics* **42** 1706–1716.

[11] Shannon, C. E. (1956) The zero-error capacity of a noisy channel. *IEEE Trans. Inform. Theory* **IT-2** 8–19.

[12] Yao, A. (1979) Some complexity questions related to distributive computing. *Proceedings 11th ACM STOC* 209–213.