On Extremal Set Partitions in Cartesian Product Spaces

RUDOLF AHLSWEDE and NING CAI

Universität Bielefeld, Fakultät für Mathematik, Postfach 100131, 33501 Bielefeld, Germany

Received 8 April 1993; revised 18 August 1993

For Paul Erdős on his 80th birthday

The partition number of a product hypergraph is introduced as the minimal size of a partition of its vertex set into sets that are edges. This number is shown to be multiplicative if all factors are graphs with all loops included.

1. Introduction

Consider \((\mathcal{V}, \mathcal{E})\), where \(\mathcal{V}\) is a finite set and \(\mathcal{E}\) is a system of subsets of \(\mathcal{V}\). For the cartesian products \(\mathcal{V}^n = \prod_i^n \mathcal{V}\) and \(\mathcal{E}^n = \prod_i^n \mathcal{E}\), let \(\pi(n)\) denote the minimal size of a partition of \(\mathcal{V}^n\) into sets that are elements of \(\mathcal{E}^n\) if a partition exists at all, otherwise \(\pi(n)\) is not defined. This is obviously exactly the case if it is so for \(n = 1\).

Whereas the packing number \(p(n)\), that is the maximal size of a system of disjoint sets from \(\mathcal{E}^n\), and the covering number \(c(n)\), that is the minimal number of sets from \(\mathcal{E}^n\) to cover \(\mathcal{V}^n\), have been studied in the literature, this seems to be not the case for the partition number \(\pi(n)\).

Obviously, \(c(n) \leq \pi(n) \leq p(n)\), if \(c(n)\) and \(\pi(n)\) are well defined. The quantity \(\lim_{n \to \infty} \frac{1}{n} \log p(n)\) is Shannon's zero error capacity [11]. Although it is known only for very few cases (see [7]), a nice formula exists for \(\lim_{n \to \infty} (1/n) \log c(n)\) (see [1, 10]).

The difficulties in analyzing \(\pi(n)\) are similar to those for \(p(n)\). For the case of graphs with edge set \(\mathcal{E}\) including all loops, we prove that \(\pi(n) = \pi(1)^n\) (Theorem 3). This result is derived from the corresponding result for complete graphs (Theorem 2) with the help of Gallai's Lemma in matching theory [6]. More general results concern products of hypergraphs with non-identical factors. Another interesting quantity is \(\mu(n)\), the maximal size of a partition of \(\mathcal{V}^n\) into sets that are elements of \(\mathcal{E}^n\) (again only hypergraphs \((\mathcal{V}, \mathcal{E})\) with a partition are considered). We also call \(\mu\) the maximal partition number. It behaves more like the packing number (see example 5). Clearly, \(\pi(n) \leq \mu(n) \leq p(n)\).

It seems to us that an understanding of these partition problems would be a significant contribution to an understanding of the basic, and seemingly simple, notion of Cartesian...
products. Another partition problem was formulated in [12]. Among the contributions to this problem, we refer the reader to [5], [9], and [12].

2. Products of complete graphs: first results

For a complete graph $G = \{V, E\}$, let $\mathcal{E}^* = E \cup \{\{v\} : v \in V\}$ and define the hypergraph $\mathcal{G}^n = \{V^n, E^n\}$, where $V^n = \Pi_1^n V$ and $E^n = \Pi_1^n E^*$.

We study the partition number $\pi(n)$, first for $\mathcal{G}^n$, and in later sections extend our results to hypergraphs, which are products of arbitrary graphs including all loops.

First we introduce the map $\sigma : E^n \to \{0, 1\}^n$, where

$$s^n = \sigma(E^n) = (\log |E_1|, \ldots, \log |E_n|).$$

As weight of $E^n$ (w($E^n$) for short), we choose the Hamming weight $w_H(s^n) = \sum_{i=1}^n s_i$. Notice that the cardinality $|E^n|$ equals $2^{w(E^n)}$.

Instead of partitions, we consider more generally a packing $\mathcal{P}$ of $\mathcal{G}^n$. We set

$$\mathcal{P}_i = \{E^n \in \mathcal{P} : w(E^n) = i\}, P_i = |\mathcal{P}_i|,$$

and call $\{P_i\}_{i=0}^n$ the weight distribution of $\mathcal{P}$.

We associate with $\mathcal{P}$ the set of shadows $\mathcal{L} \subset 2^n$ defined by

$$\mathcal{L} = \{E^n \in \mathcal{E} : E^n \subset F^n \text{ for some } F^n \in \mathcal{P}\},$$

and its level sets

$$\mathcal{L}_i = \{E^n \in \mathcal{L} : w(E^n) = i\}, 0 \leq i \leq n.$$

It is convenient to write $Q_i = |\mathcal{L}_i|$. $\{Q_i\}_{i=0}^n$ is the weight distribution of $\mathcal{L} = \text{shad}(\mathcal{P})$.

First we establish some simple connections between these weight distributions.

**Lemma 1.** For a packing $\mathcal{P}$ of $\mathcal{G}^n$

$$\sum_{i=k}^{n} 2^{i-k}\binom{i}{k} P_i = Q_k. \quad (2.5)$$

**Proof.** Consider any edge $E^n$ with weight $w(E^n) = i \geq k$. There are exactly $2^{i-k}\binom{i}{k}$ edges contained in $E^n$ with weight $k$. Therefore we have always

$$\sum_{i=k}^{n} 2^{i-k}\binom{i}{k} P_i \geq Q_k. \quad (2.6)$$

\[\square\]

**Lemma 2.** For a packing $\mathcal{P}$ of $\mathcal{G}^n$

$$|\mathcal{P}| = \sum_{i=0}^{n} P_i = \sum_{k=0}^{n} (-1)^k Q_k. \quad (2.7)$$
Proof. An edge $E^n \in \mathcal{P}_i$ contributes to $\sum_{k=0}^{n}(-1)^{k}Q_k$ the amount
\[
\sum_{k=0}^{i}(-1)^{k}2^{1-k}\binom{i}{k} = (2-1)^i = 1.
\]

Lemma 3. For a packing $\mathcal{P}$ of $\mathcal{C}^n$
\[
P_0 = \sum_{k=0}^{n}(-1)^{k}2^{k}Q_k
\]
and if in addition $\mathcal{P}$ is a partition and $S = |\mathcal{V}|$ is odd,
\[
\sum_{k=0}^{n}(-1)^{k}2^{k}Q_k - 1 \geq 0.
\]
Proof. An edge $E^n \in \mathcal{P}_i$ contributes to $\sum_{k=0}^{n}(-1)^{k}2^{k}Q_k$ the amount
\[
\sum_{k=0}^{i}(-1)^{k}2^{k}2^{1-k}\binom{i}{k} = 2^i(1 - 1)^i,
\]
which equals 1, if $i = 0$, and 0, otherwise.
Therefore (2.8) holds.
Furthermore, if $S$ is odd, then so is $S^n$ and there must be an edge in the partition of odd size, that is, $P_0 \geq 1$ or, equivalently, by (2.8), (2.9) must hold.

Remark 1. The last two Lemmas can be derived more systematically from Lemma 1 by M"obius Inversion. Here this machinery can be avoided, but we need it for the more abstract setting of [4].

3. Products of complete graphs: the main results

We shall now exploit Lemma 3 by applying it to classes of subhypergraphs, which we now define. For any $I \subset \{1, 2, \ldots, n\}$ and any specification $(v^j)_{j \in I}$, where $v_j \in \mathcal{V}_i$, we set
\[
\mathcal{C}^n(I, (v^j)_{j \in I}) = \left(\prod_{i=1}^{n} U_i, \prod_{i=1}^{n} F_i\right) = \left(\mathcal{U}^n, \mathcal{F}^n\right),
\]
where
\[
\mathcal{U}_i = \begin{cases} \mathcal{V}_i & \text{and } F_i = \begin{cases} \mathcal{E}_i & \text{for } i \in I \\ \{v_i\} & \text{for } i \in I^c \end{cases} \end{cases}
\]
Clearly, for a partition $\mathcal{P}$ of $\mathcal{C}^n$ and $\mathcal{Q} = \text{shad}\mathcal{P}$, the set $\mathcal{Q}(I, (v^j)_{j \in I}) = \mathcal{Q} \cap \mathcal{F}^n$ is a downset, and the map
\[
\psi : \mathcal{F}^n \rightarrow \prod_{i \in I} \mathcal{E}_i, \psi \left(\prod_{i=1}^{n} E_i\right) = \prod_{i \in I} E_i
\]
is a bijection.
Write \( \mathcal{F} = \psi(\mathcal{F} \cap \mathcal{F}^n) \) and let \( \mathcal{F}_i \) count the members of \( \mathcal{F} \) of weight \( i \). Since \( \mathcal{F} \) is a downset in \( \prod_{i \in I} \mathcal{F}_i \) and its maximal elements form a partition of \( \prod_{i \in I} \mathcal{V}_i \), we know that \( \mathcal{F}_0 = S^m \). This fact and Lemma 3 yield

\[
S^m + \sum_{k=1}^{m} (-1)^k 2^k S^k - 1 \geq 0. \tag{3.4}
\]

This is the key to the proof of the following important result.

**Theorem 1.** For a partition \( \mathcal{P} \) of \( \mathcal{F}^n = (\mathcal{V}^n, \mathcal{F}^n) \) with \( \mathcal{V}^n = \prod_{i=1}^{n} \mathcal{V}_i \), \( |\mathcal{V}_i| = S \) for \( i = 1, 2, \ldots, n \), the weight distribution \( (Q_k)_{k=0}^{n} \) of \( Q = \text{shad} \mathcal{P} \) satisfies, for \( 1 \leq m \leq n \),

\[
\binom{n}{m} S^n + \sum_{k=1}^{m} (-1)^k \binom{n-k}{m-k} 2^k Q_k - \binom{n}{m} S^{n-m} \geq 0. \tag{3.5}
\]

**Proof.** The map \( \psi \) preserves inclusions and weights. The total number of pairs \( (I, (v_i)_{i \in I}) \) with \( |I| = m \) equals \( \binom{n}{m} S^{n-m} \). Moreover, each \( E^n \in \mathcal{F} \) with \( w(E^n) = k \) is contained in exactly \( \binom{n-k}{m-k} \) sets of the form \( \mathcal{F}(I, (v_i)_{i \in I}) \) and thus for the sets of weight \( k \)

\[
\binom{n-k}{m-k} Q_k = \sum_{(I, (v_i)_{i \in I}), |I| = m} |\mathcal{F}(I, (v_i)_{i \in I})|. \tag{3.6}
\]

We have one equation of the form (3.4) for each pair \( (I, (v_i)_{i \in I}) \). Summation of their left-hand sides gives, therefore,

\[
\binom{n}{m} S^{n-m} \cdot S^n + \sum_{k=1}^{m} (-1)^k 2^k \binom{n-k}{m-k} Q_k - \binom{n}{m} S^{n-m} \geq 0
\]

and hence (3.5).

Now comes the harvest.

**Theorem 2.** For a partition \( \mathcal{P} \) of \( \mathcal{F}^n \)

\[
|\mathcal{P}| \geq \left[ \frac{S^n}{2^n} \right] .
\]

**Proof.** Since \( |E^n| \leq 2^n \), obviously \( |\mathcal{P}| \geq S^n/2^n \), and for \( S = 2 \alpha \) even, the result obviously holds. Now let \( S = 2 \alpha + 1 \).

Summing the left-hand side expressions in (3.5) for \( m = 1, 2, \ldots, n \) results in

\[
\sum_{m=1}^{n} \binom{n}{m} S^n + \sum_{m=1}^{n} \sum_{k=1}^{m} (-1)^k \binom{n-k}{m-k} 2^k Q_k - \sum_{m=1}^{n} \binom{n}{m} S^{n-m} \geq 0,
\]
or in
\[(2^n - 1)S^n + \sum_{k=1}^{n} (-1)^k 2^k Q_k \sum_{m=k}^{n} \binom{n-k}{m-k} - [(S+1)^n - S^n] \geq 0.\]

This is equivalent to
\[2^n \cdot [S^n + \sum_{k=1}^{n} (-1)^k Q_k] - (S+1)^n \geq 0.\]

As \(Q_0 = S^n\), we conclude, with Lemma 2,
\[P \geq (S+1)^n \cdot 2^{-n} = \left[\frac{S^n}{2}\right], \text{ if } S \text{ is odd.} \]

4. Non-identical factors: a generalization

We now consider hypergraphs \(\mathcal{G}^n\) with vertex sets \(\mathcal{V}^n = \prod_{t=1}^{n} \mathcal{V}_t\) and edge sets \(\mathcal{E}^n = \prod_{t=1}^{n} \mathcal{E}_t\), where the \(\mathcal{V}_t\)'s are finite sets of not necessarily equal cardinalities \(S_t\). The factors \(\mathcal{E}_t\) are such that \((\mathcal{V}_t, \mathcal{E}_t)\) is a complete graph with all loops included. We shall write, with positive integers \(\alpha_t\),
\[|\mathcal{V}_t| = 2\alpha_t + \varepsilon_t, \varepsilon_t \in \{0,1\}. \quad (4.1)\]

Inspection shows that the sizes of factors do not affect the proofs of Lemmas 1 and 2. Also (2.8) in Lemma 2 holds and since \(P_0 \geq 1\), if \(\varepsilon_t = 1\) for \(t = 1, 2, \ldots, n\), we can generalize (2.9) to
\[\sum_{k=0}^{n} (-1)^k 2^k Q_k - \prod_{k=1}^{n} \varepsilon_k \geq 0. \quad (4.2)\]

Theorem 1 in Section 3 generalizes to

**Theorem 1'.** For a partition \(\mathcal{P}\) of \(\mathcal{G}^n\)

\[\binom{n}{m} \prod_{i=1}^{n} S_i + \sum_{k=1}^{m} (-1)^k \binom{n-k}{m-k} 2^k Q_k - \sum_{I: |I|=m} \prod_{i \in I} \varepsilon_i \prod_{j \in I^c} S_j \geq 0. \quad (4.3)\]

**Proof.** (Sketch) In the proof of Theorem 1, replace \(S^m\) by \(\prod_{i \in I} S_i\) and inequality (3.4) by
\[\prod_{i \in I} S_i + \sum_{k=1}^{n} (-1)^k 2^k Q_k - \prod_{i \in I} \varepsilon_i \geq 0. \quad (4.4)\]

**Theorem 2'.** For a partition \(\mathcal{P}\) of \(\mathcal{G}^n\)

\[|\mathcal{P}| \geq \prod_{i=1}^{n} \left\lceil \frac{S_i}{2} \right\rceil. \quad (4.5)\]
Proof. Summing the expressions on the left-hand side in (4.3) for \( m = 1, 2, \ldots, n \) results in

\[
0 \leq \sum_{m=1}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) \prod_{i=1}^{n} S_i + \sum_{m=1}^{n} \sum_{k=1}^{m} \binom{n-k}{m-k}(-1)^k 2^k Q_k - \sum_{m=1}^{n} \sum_{|I|=m} \prod_{i \in I} e_i \prod_{j \in I^c} S_j \\
= (2^n - 1) \prod_{i=1}^{n} S_i + \sum_{k=1}^{n} (-1)^k 2^k Q_k \sum_{m=k}^{n} \binom{n-k}{m-k} - \sum_{\phi \neq I} \prod_{i \in I} e_i \prod_{j \in I^c} S_j \\
= 2^n \left[ \prod_{i=1}^{n} S_i + \sum_{k=1}^{n} (-1)^k Q_k \right] - \sum_{I} \prod_{i \in I} e_i \prod_{j \in I^c} S_j
\]

or

\[
|\mathcal{P}| \geq 2^{-n} \sum_{I} \prod_{i \in I} e_i \prod_{j \in I^c} S_j. \tag{4.6}
\]

We evaluate the expression on the right-hand side by introducing \( J = \{ \ell : 1 \leq \ell \leq n, \ v_\ell = 1 \} \) and \( I^* = J \setminus I \). Then

\[
\sum_{I} \prod_{i \in I} e_i \prod_{j \in I^*} S_j = \sum_{I \subset J} \prod_{i \in I} S_i \cdot \prod_{j \in I^*} S_j
\]

\[
= \prod_{j \in J} (S_j + 1) \cdot \prod_{j \in I^*} S_j = \prod_{j=1}^{n} (S_j + e_j) \quad \text{and (4.5) follows.} \quad \qed
\]

Corollary 1. The partition number \( \pi(\mathfrak{G}^m) \) equals \( \prod_{j=1}^{n} \left[ \frac{S_j}{2} \right] \).

Proof. The partition number of \( (\mathcal{V}, \mathcal{E}) \) is \( \left[ \frac{S_j}{2} \right] \). Take a product of optimal partitions for the factors. This construction gives the lower bound in Theorem 2'. \quad \qed

5. Products of general graphs

We assume now that the factors \( \mathcal{G}_t = (\mathcal{V}_t, \mathcal{E}_t) \) \( (t = 1, 2, \ldots, n) \) are arbitrary finite graphs with all loops included.

Obviously, we have for the partition number

\[
\pi(\mathcal{G}_t) = |\mathcal{V}_t| - v(\mathcal{G}_t), \tag{5.1}
\]

where \( v(\mathcal{G}_t) \) is the matching number of \( \mathcal{G}_t \).

Theorem 3. For the hypergraph product \( \mathcal{H}^n = \mathcal{G}_1 \times \ldots \times \mathcal{G}_n \)

\[
\pi(\mathcal{H}^n) = \prod_{i=1}^{n} \pi(\mathcal{G}_t). \tag{5.2}
\]

\[\text{\textcopyright 2023 R. Ahlswede and N. Cai} \]
Here only the inequality
\[ \pi(H^n) \geq \prod_{t=1}^{n} \pi(G_t) \] (5.3)
is non-trivial. We make use of a well-known result from matching theory.

**Gallai's Lemma.** ([6] or [8] page 89) If a graph \( G = (V, E) \) is connected and, for all \( v \in V \), \( v(G - v) = v(G) \), then \( G \) is factor-critical, that is, for all \( v \in V \), \( G - v \) has a perfect matching.

**Proof of 5.3.** For every \( t \in \{1, 2, \ldots, n\} \) we modify \( G_t \) as follows: remove any vertex \( v \in V_t \) with \( v(G_t - v) < v(G_t) \) and reiterate this until a graph \( G^*_t \) with \( v(G^*_t - v) = v(G^*_t) \) for all \( v \in V^*_t \) is obtained.

Notice that (5.1) ensures that
\[ \pi(G_t) = \pi(G^*_t). \] (5.4)
Denote the set of connected components of \( G^*_t \) by \( \{G^*_t(j)\}_{j \in L_t} \). Clearly,
\[ \pi(G^*_t) = \sum_{j \in L_t} \pi(G^*_t(j)). \] (5.5)
Moreover, by Gallai's Lemma each component \( G^*_t(j) \) has a vertex set \( V^*_t(j) \) of odd size and
\[ v(G^*_t(j)) = |V^*_t(j)| - 1 \equiv \alpha_t^j, \text{ say.} \]
Thus,
\[ \pi(G^*_t) = \sum_j (\alpha_t^j + 1). \] (5.6)
Now, for \( H^n = \prod_t G^*_t \) we have
\[ \pi(H^n) \geq \pi(H^n), \] (5.7)
because the modifications described above transform a partition of \( H^n \) into a partition of \( H^n \) with no more parts.

Finally, by Theorem 2', we have for the product \( \mathcal{C}^n \) of complete graphs with vertex sets \( V^*_t(j) \) that
\[ \pi(G^*_1(j_1) \times \ldots \times G^*_n(j_n)) = \pi(G^*_t(j_1) \times \ldots \times G^*_n(j_n)). \] (5.8)
Therefore,
\[ \pi(H^n) = \sum_{j_1 \in L_1, \ldots, j_n \in L_n} \pi(G^*_1(j_1) \times \ldots \times G^*_n(j_n)) \geq \sum_{(j_1, \ldots, j_n)} (\alpha_1^{j_1} + 1) \ldots (\alpha_n^{j_n} + 1) \]
This and (5.7) imply (5.3).

6. Examples for deviation from multiplicative behaviour

First we give two examples of product hypergraphs $\mathcal{H} \times \mathcal{H}'$ for which the partition number $\pi$ is not multiplicative in the factors. They are due to K.-U. Koschnick.

Example 1.

$\mathcal{V}_1 = \{0, 1, 2, \ldots, 6\}$, $\mathcal{E}_1 = \{E \subseteq \mathcal{V}_1 : |E| \in \{1, 4\}\}$.

Clearly, $\pi(\mathcal{H}_1) = 4$ and the partition

$$
\{\{i\} \times \{0, 1, 2, 3\} : i = 0, 1, 2\} \cup \{\{i\} \times \{3, 4, 5, 6\} : i = 4, 5, 6\}
\cup \{\{0, 1, 2, 3\} \times \{j\} : j = 4, 5, 6\}
\cup \{\{3, 4, 5, 6\} \times \{j\} : j = 0, 1, 2\}
\cup \{\{3\} \times \{3\}\}
$$

has 13 members. Therefore

$$
\pi(\mathcal{H}_1 \times \mathcal{H}_1) \leq 13 < \pi(\mathcal{H}_1) \pi(\mathcal{H}_1) = 16.
$$

(6.1)

While this example seems to be the smallest possible for identical factors, one can do better with non-identical factors:

$\mathcal{H}_1 \times \mathcal{H}'_1$, where $\mathcal{V}_1 = \{0, 1, 2, 3, 4\}$ and $\mathcal{E}_1 = \{E \subseteq \mathcal{V}_1 : |E| \in \{1, 3\}\}$.

Here, by a similar construction, $\pi(\mathcal{H}_1 \times \mathcal{H}'_1) \leq 11$, whereas $\pi(\mathcal{H}_1) \cdot \pi(\mathcal{H}'_1) = 4 \cdot 3 = 12$.

Example 2. Since $\pi$ is multiplicative for graphs, one may wonder whether it is multiplicative if one factor is a graph.

Consider $G = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{0, 1, \ldots, 4\}$ and $\mathcal{E} = \{(i, i+1 \text{ mod } 5) : i = 0, 1, \ldots, 4\} \cup \{i : 0 \leq i \leq 4\}$, that is, the pentagon with all loops.

Define $\mathcal{H}' = (\mathcal{V}', \mathcal{E}')$ with $\mathcal{V}' = \{1, 2, \ldots, 14\}$ and $\mathcal{E}' = \{E \subseteq \mathcal{V}' : |E| \in \{1, 9\}\}$.

Notice that $\pi(G) = 3$, $\pi(\mathcal{H}') = 7$, and that the following construction ensures $\pi(G \times \mathcal{H}') \leq 20 < 21 = \pi(G) \cdot \pi(\mathcal{H}')$:

$$
\{\{i\} \times \{j + k \text{ mod } 14 : 0 \leq k \leq 8\} : (i, j) \in \{(0, 0), (1, 3), (2, 6), (3, 9), (4, 12)\}\}
\cup \{\{1, 2\} \times \{j\} : j = 0, 1, 2\} \cup \{\{2, 3\} \times \{j\} : j = 2, 3, 5\}
\cup \{\{3, 4\} \times \{j\} : j = 6, 7, 8\}
\cup \{\{4, 0\} \times \{j\} : j = 9, 10, 11\}
\cup \{\{0, 1\} \times \{j\} : j = 12, 13, 14\}
$$

is a set of $5 + 5 \cdot 3 = 20$ edges partitioning $\mathcal{V} \times \mathcal{V}'$. 
To help orient the reader, we add three examples, which demonstrate that the covering number $c$, the packing number $p$ and the maximal partition number $\mu$ are not multiplicative in the factors either.

**Example 3.** $\mathcal{V}_3 = \{0,1,2\}$, $\mathcal{E}_3 = \{E \subseteq \mathcal{V} : |E| = 2\}$

We have

$$3 = c(\mathcal{V}_3 \times \mathcal{V}_3) \neq c(\mathcal{V}_3) \cdot c(\mathcal{V}_3) = 4,$$

because $\mathcal{E} = \{0,1\} \times \{0,1\}, \{0,2\} \times \{0,2\}, \{1,2\} \times \{1,2\}\}$ covers $\mathcal{V}_3 \times \mathcal{V}_3$ and there is no covering with 2 edges.

This is the smallest example in terms of the number of vertices.

**Remark 2.** Quite generally, even in the case of non-identical factors $\mathcal{X}_t = (\mathcal{V}_t, \mathcal{E}_t), t \in \mathbb{N}$, with $\max_t |\mathcal{E}_t| < \infty$, the asymptotic behaviour of $c(n)$ is known [1]:

$$\lim_{n \to \infty} \frac{1}{n} \left( \log c(n) - \sum_{t=1}^{n} \log \left( \max_{\mathcal{E} \in \text{Prob}(\mathcal{E}_t)} \min_{E \in \mathcal{E}} \sum_{v \in E} 1_{E}(v)q_{E} \right)^{-1} \right) = 0,$$

where $\text{Prob}(\mathcal{E}_t)$ is the set of all probability distributions on $\mathcal{E}$, $q_{E}$ is the probability of $E$ under $q$ and $1_{E}$ is the indicator function of the set $E$.

**Example 4.** $\mathcal{V}_4 = \{0,1,2,3,4\}$, $\mathcal{E}_4 = \{\{x, x+1 \mod 5\} : x \in \mathcal{V}_4\}$.

Here we have

$$5 = p(\mathcal{V}_4 \times \mathcal{V}_4) \neq p(\mathcal{V}_4)p(\mathcal{V}_4) = 4.$$

It was shown in [11] that this is the smallest example in the previous sense. Notice that it is bigger than the previous one.

**Example 5.** To avoid heavy notation, we will write $\mathcal{H}_5 = (\mathcal{V}_5, \mathcal{E}_5)$ without an index as $\mathcal{H} = (\mathcal{V}, \mathcal{E})$. It is made up of the 5 vertex sets

$$\mathcal{V}_i = \{x_{ij} : j = 1,2,\ldots, m\}, 3 \leq m(i = 0,1,2,\ldots,4),$$

the 6 edge sets

$$\mathcal{E}_i = \{(x_{ij}, x_{i+1 \mod 5}, j) : j = 1,2,\ldots, m\}(i = 0,1,2,\ldots,4),$$

and $\{\mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4\}$. Thus

$$\mathcal{V} = \bigcup_{i=0}^{4} \mathcal{W}_i, \mathcal{E} = \{\mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4\} \cup \bigcup_{i=0}^{4} \mathcal{E}_i.$$

A look at the pentagon with vertex set $\{x_{01}, x_{11}, x_{21}, x_{31}, x_{41}\}$ shows that a partition of $\mathcal{V}$ must contain at least one of the edges $\mathcal{W}_i$ as a member. On the other hand, the vertices $\mathcal{V} \setminus \mathcal{W}_i$ have a maximal partition of size $2m$. Therefore we have shown that $\mu(\mathcal{V}) = 2m + 1$. We shall next consider $\mu(\mathcal{H} \times \mathcal{H})$. For this we introduce the superedges

$$\mathcal{G}_i = \mathcal{W}_i \cup \mathcal{W}_{i+1 \mod 5}(i = 0,1,\ldots,4)$$

in $\mathcal{H}$, and the superedges $\mathcal{G}_i \times \mathcal{G}_j(i,j = 0,1,\ldots,4)$ in $\mathcal{H} \times \mathcal{H}$. Whereas $\mathcal{G}_i$ can be partitioned into $m$ edges, they can be partitioned into $m^2$ edges.
First we divide $\mathcal{V} \times \mathcal{V}$ into 25 parts $\{\mathcal{W}_i \times \mathcal{W}_{i'} : i, i' = 0, 1, \ldots, 4\}$. Then we pack 5 superedges (as in Shannon's construction) into $\mathcal{V} \times \mathcal{V}$. They cover 20 parts, and the remaining 5 parts are packed with 5 edges of type $\mathcal{W}_i \times \mathcal{W}_i$. Finally, we partition the 5 superedges into the edges of $\mathcal{H} \times \mathcal{H}$. Thus we obtain a desired partition with $5 + 5m^2$ edges. Notice that $\mu(\mathcal{H} \times \mathcal{H}) \geq 5 + 5m^2 > (2m + 1)^2 = \mu(\mathcal{H})^2$ for $m \geq 3$. The smallest example in this class has 15 vertices.

**Remark 3.** The construction was based on the pentagon. Its vertices were replaced by sets of vertices $\mathcal{W}_i$ with a numbering. The vertices with the same number in the $\mathcal{W}_i$'s form a pentagon. Thus we obtained $m = |\mathcal{W}_i|$ many pentagons. Then we added the $\mathcal{W}_i$ as further edges. Finally we used the superedges to mimic the original small edges. We can make this construction starting with any hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$. If it has the property $p(\mathcal{H})^2 < p(\mathcal{H} \times \mathcal{H})$, then for $m$ large enough our construction gives an associated hypergraph for which $\mu$ is not multiplicative.

### 7. Acknowledgement

The authors are very much indebted to Klaus-Uwe Koschnick for constructing beautiful examples.

### References


