

# On Partitioning the $n$ -Cube into Sets with Mutual Distance 1

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## Abstract

We consider a problem mentioned in [1], which is in partitioning the  $n$ -cube in as many sets as possible, such that two different sets always have distance one.

## 1 Introduction and approach

Assume that the vertices of the  $n$ -cube are partitioned in such a way that the Hamming distance between each pair of the subsets in the partition is 1. What is the maximal number of such subsets? Denoting this number by  $m(n)$ , one can easily check the following table:

$n$ :	0	1	2	3	4
$m(n)$ :	1	2	3	4	8

Only the case  $n = 4$  may raise some difficulties. In this case the 4-cube must be partitioned into 8 sets each of size 2, with the required properties. Below we present one such partition  $\{A_i : i = 1, \dots, 8\}$ , representing the vertices of the 4-cube as binary strings:

$A_1$	$A_2$	$A_3$	$A_4$	$A_5$	$A_6$	$A_7$	$A_8$
0000	1000	0100	0010	0001	0011	0110	0101
1111	0111	1010	1001	1100	1110	1101	1011

**Proposition 1**  $m(n) \leq \sqrt{n2^n} + 1$ .

*Proof.*

Since the  $n$ -cube has  $n2^{n-1}$  edges and each edge can only realize distance one between two sets, it immediately follows that  $\binom{m}{2} \leq n2^{n-1}$ . ■

**Proposition 2**  $m(n) \leq \lfloor \frac{2^n}{m(n)} \rfloor n + 1$ .

*Proof.*

If one has a partition of the  $n$ -cube into  $m$  sets, then one of the sets has at most  $\lfloor 2^n/m \rfloor$  elements and hence can have distance one to at most  $\lfloor 2^n/m \rfloor n$  other sets. ■

The reason for including this improved upper bound is that it gives the correct values for  $n = 1, 2, \dots, 4$ .

Now we give a simple lower bound, which later will be improved.

**Proposition 3**  $m(n) \geq \sqrt{2^n}$ .

*Proof.*

Consider the  $2^{n-1}$  edges of the  $n$ -cube having a fixed direction, i.e., which connect vertices having some (say the first)  $n - 1$  entries equal. These edges are disjoint and partition the vertices of the  $n$ -cube. Now if  $\binom{m}{2} \leq 2^{n-1}$  there are enough edges to create distance one between all the  $m$  sets, because we can label the endpoints of the edges independently of one another.

To be specific, we can consider any system of disjoint sets  $B_i \subseteq \{0, 1\}^{n-1}$  of cardinality  $|B_i| = m - i$  ( $i = 1, \dots, m - 1$ ), label the vertices in  $B_i \times \{0\}$  (in anyway) with  $i + 1, \dots, m$  and in  $B_i \times \{1\}$  with  $i$ . The remaining vertices in the  $n$ -cube can be labeled by any number from 1 to  $m$ . ■

**Remark 1** *Instead of the partition above one can also partition the vertices of the  $n$ -cube in squares (that is 2-cubes). Now take  $m/2$  squares and label a pair of diagonal vertices for all of them with 1 and 2. The other pairs of diagonal vertices of these squares we label with  $(1, 2), (3, 4), \dots, (m - 1, m)$ . At this step we have that the sets given by labels 1 and 2 have distance 1 with all other sets. Further, take another  $(m - 2)/2$  squares and label one pair of diagonal vertices of them with 3 and 4, and label similarly to above the other diagonal vertices with  $(3, 4), \dots, (m - 1, m)$ . Continuing in such a way one can get a lower bound  $m(n) \geq \sqrt{2^{n+1}}$ .*

For the general case we try to imitate this process: we partition the  $n$ -cube in  $2^{n-k}$   $k$ -cubes and try to partition the  $k$ -cube in two parts. One part will contain a fixed set of consecutive labels, starting with label  $l$  say, distributed in some way, and this will be so for a certain number of  $k$ -cubes, and the other parts will contain all labels  $l, \dots, m$  if we take these  $k$ -cubes all together. The trick is to find a way to partition the  $k$ -cube in such a way that this works and essentially produces the only edge between any pair  $(i, j)$  of subsets. As is easy to see for  $k = 3$  this is not possible in general, but it can be done if  $k$  is a power of 2. This leads us to the following bound:

**Theorem 1**  $m(n) \geq \frac{\sqrt{2}}{2} \sqrt{n 2^n}$ .

## 2 The construction

Now we use the dual form of our problem. For a given number  $m$  of subsets we want to find the minimal dimension  $n$  of the unit cube, in which the required partition is possible.

Consider first the unit cube of dimension  $k = 2^a$ . Partition this  $k$ -cube in two parts, the words of even weight, and the words of odd weight. Let  $H$  be the (extended) Hamming code of length  $2^k$ . So every word in  $H$  has even weight,  $H$  is a linear code of dimension  $k - 1 - a$  and minimum distance 4.

Now partition the set of words of even weight in cosets of  $H$ . It follows that every word of odd weight has a unique neighbor in each even coset of  $H$ . Indeed, if some such word  $\alpha$  has at least two neighbors  $\beta, \gamma$  in some coset, then the Hamming distance between  $\beta$  and  $\gamma$  is 2, but the minimum distance in the code and all its cosets is 4.

To make the description a little bit easier, we make our construction in two steps. At the first step we produce for every pair  $(i, j)$  of subsets two edges connecting them. At the second step we improve the construction by producing almost each pair  $(i, j)$  once.

Let  $m = 2^{2^a-1}$  and partition the set  $\{1, \dots, m\}$  in  $2^{2^a-a-1}$  groups of consecutive numbers. Therefore each group is of size  $2^a$ , which equals the number of even cosets of  $H$ . Now for the  $l$ -th group we take a  $k$ -cube and label its points as follows. Every even coset of  $H$  gets its own fixed number from the  $l$ -th group. The words of odd weight all get different labels  $1, \dots, m$ . Since the number of such words is exactly  $m$ , this is precisely possible. We now have realized connections between all pairs  $(i, j)$  of subsets with  $i$  in the  $l$ -th group and  $j$  arbitrary. Next we do the same for each  $l$  ( $l = 1, 2, \dots, 2^{2^a-a-1}$ ).

The total number of  $2^a$ -cubes we use in this process equals the number of groups, i.e.,  $2^{2^a-a-1}$ . So this produces a partition of the cube of dimension  $n = 2^{a+1} - a - 1$  in  $m = 2^{2^a-1}$  sets and for  $m/\sqrt{n2^n}$  we get

$$\frac{2^{2^a-1}}{\sqrt{(2^{a+1} - (a + 1))2^{2^{a+1}-(a+1)}}},$$

and this tends to  $1/2$  as  $a \rightarrow \infty$ .

What is not so good in this construction is that we produce exactly two edges between each pair of subsets, while if we like to reach the upper bound  $\sqrt{n2^n}$ , almost each pair of them should be linked by just one edge. To achieve that, choose  $m = b \cdot 2^{2^a-1}$ . Partition again the set  $\{1, \dots, m\}$  in  $b$  hypergroups of size  $2^{2^a-1}$  of consecutive numbers. Further, partition each hypergroup in  $2^{2^a-a-1}$  groups of size  $2^a$  of consecutive numbers.

Consider the first hypergroup now. For the group with number  $l$  ( $l = 1, \dots, 2^{2^a-a-1}$ ) we take  $b$  subcubes of dimension  $2^a$  and label their points as follows. Every even coset of  $H$  gets its own fixed number from the  $l$ -th group as above. The words of odd weight of the  $p$ -th subcube ( $p = 1, \dots, b$ ) get different labels from the  $p$ -th hypergroup. Now we have realized connections between pairs  $(i, j)$  of subsets with  $i$  in the first hypergroup and  $j$  arbitrary.

In the next step we consider the second hypergroup, take for each group of it  $b - 1$  subcubes of dimension  $2^a$ , and repeat the procedure in the last paragraph with the only difference that we forget about the first hypergroup now. So that way we produce edges of type  $(i, j)$  with  $i$  in the first hypergroup and  $j$  in the second only once.

In general we consider the hypergroup with number  $q$  ( $q = 1, \dots, b$ ), take for each group of it exactly  $b - q + 1$  subcubes of dimension  $2^a$  and repeat with them the procedure above, labeling the odd weight vertices of them with numbers from the hypergroup with numbers  $q, q + 1, \dots, b$ .

In such a way we get the only edge between subsets  $(i, j)$  ( $i < j$ ) with  $i, j$  taken from different hypergroups, and exactly two edges of type  $(i, j)$ , if  $i, j$  belong to the same hypergroup. Therefore, when the number of hypergroups is large enough, we realized the only edge between almost all pairs of subsets.

The number of  $2^a$ -cubes we used now equals

$$2^{2^a - a - 1} \cdot (b + (b - 1) + \dots + 1) = b(b + 1)/2 \cdot 2^{2^a - a - 1}.$$

Let  $b = 2^c - 1$ , then  $b(b - 1)/2 < 2^{2^c - 1}$ . Therefore, we have a partition of the unit cube of dimension  $2^{a+1} - a - 1 + 2c - 1$  into  $(2^c - 1) \cdot 2^{2^1 - a - 1}$  sets, and the ratio  $m/\sqrt{n2^n}$  converges to  $\sqrt{2}/2$  as  $c \rightarrow \infty$ ,  $a \rightarrow \infty$ ,  $c/a \rightarrow 0$ .

### 3 Final remarks

The problem we considered above for the hypercube can be stated for graphs in general. So for a graph  $G$  the problem is to determine  $m(G)$  = the maximal number of sets in a partitioning of the vertex set of  $G$  with the property that between two different sets there always is at least one edge. If  $e = e(G)$  is the number of edges of  $G$ , then as in Proposition 1 we have

$$\binom{m}{2} \leq e$$

or roughly  $m < \sqrt{2e}$ . Now we have shown that for the hypercube this trivial upper bound is the correct value up to a constant factor, but the question remains how special this property is. Using standard probabilistic arguments it is not hard to see that

$$\liminf \frac{m(G)}{\sqrt{2e(G)}} = 0,$$

but it would still be interesting to have an explicit family of graphs  $G_i$ ,  $i = 0, 1, 2, \dots$  with  $\lim_{i \rightarrow \infty} m(G_i)/\sqrt{2e(G_i)} = 0$ . We conjecture that this is true for the Paley graphs.

### References

- [1] R.Ahlswede, N.Cai, Z.Zhang *Higher level extremal problems*, Preprint 92-031 of SFB 343 "Discrete Strukturen in der Mathematik", submitted to *Combinatorica*.