A Recursive Bound for the Number of Complete K-Subgraphs of a Graph

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Abstract

The following inequality was conceived as a tool in determining coloring numbers in the sense of Ahlswede, Cai, Zhang ([1]), but developed into something of a seemingly basic nature.

Theorem For any graph $G = (\Omega_n, \mathcal{E})$ with $n$ vertices let $T_k$ be the number of complete $k$-subgraphs of $G$. Then for $k \geq 2$

$$T_k \geq \frac{T_{k-1}}{k \cdot n} [2(k-1)|\mathcal{E}| - (k-2)n^2].$$

(1)

Proof of the Theorem stated in the abstract. By its definition we have $T_1 = n$. We show (1) by induction on $k$.

For $k = 2$ $T_2 = |\mathcal{E}| = \frac{n}{2} n [2|\mathcal{E}|] = |\mathcal{E}|$, so (1) holds even with equality. For the induction step from $k - 1$ to $k$ we need some notation.

$V(m)$ denotes a set with $m$ vertices and $T_m$ stands for the set of all those sets, which are the vertex set of a complete $m$-subgraph. We also set

$$\mathcal{E}_v = \{v' : (v, v') \in \mathcal{E}\},$$

$$\mathcal{E}_{V(m)} = \cap_{v \in V(m)} \mathcal{E}_v,$$

and start now with

$$T_k = \frac{1}{k} \sum_{V(k-1) \in T_{k-1}} |\mathcal{E}_{V(k-1)}|.$$

Next we bound $|\mathcal{E}_{V(k-1)}|$ from below with the help of the identity

$$\left| \bigcup_{V(k-2) \subseteq V(k-1)} \mathcal{E}_{V(k-2)} \right| = \sum_{V(k-2) \subseteq V(k-1)} |\mathcal{E}_{V(k-2)}| - (k-2)|\mathcal{E}_{V(k-1)}|,$$

(6)

which holds, because vertices from the union which are counted more than once in the sum are actually counted $k-1$ times and they are exactly the vertices in $\mathcal{E}_{V(k-1)}$.

Since the union has a cardinality not exceeding $n$, we get

$$|\mathcal{E}_{V(k-1)}| \geq \frac{1}{k-2} \left( \sum_{V(k-2) \subseteq V(k-1)} |\mathcal{E}_{V(k-2)}| - n \right).$$

(7)
Substituting this in (5) yields

\[
k(k - 2)T_k \geq \sum_{V(k-1) \in T_{k-1}} \left( \sum_{V(k-2) \subseteq V(k-1)} |E_{V(k-2)}| - n \right)
\]
\[
= \sum_{V(k-2) \in T_{k-2}} \sum_{V(k-1) \in T_{k-1}(V(k-2))} |E_{V(k-2)}| - n|T_{k-1}|
\]
\[
= \sum_{V(k-2) \in T_{k-2}} |T_{k-1}(V(k-2))|^2 - nT_{k-1}
\]
\[
\geq T_{k-2} \left( \frac{(k - 1)T_{k-1}}{T_{k-2}} \right)^2 - nT_{k-1} \quad \text{(by convexity of } x^2 \text{)}
\]
\[
= \frac{T_{k-1}}{T_{k-2}}((k - 1)^2T_{k-1} - nT_{k-2})
\]
\[
\geq \frac{T_{k-1}}{T_{k-2}} \left( (k - 1)\frac{T_{k-2}}{n}(2(k - 2)|E| - (k - 3)n^2) - nT_{k-2} \right)
\]
\[
= \frac{T_{k-1}}{n} (2(k - 1)(k - 2)|E| - ((k - 1)(k - 3) + 1)n^2)
\]

and therefore (1).

The following consequence is useful.

**Corollary** If for some \( \alpha > 0 \) \( |E| \geq \frac{k-1}{2k}n^2 + \alpha n^2 \), then

\[
T_{k+1} \geq \alpha^k n^{k+1}.
\]  \( \text{(8)} \)

**Proof:** Since \( \frac{k-1}{2k} \geq \frac{\ell-1}{2\ell} \) for \( \ell = 1, 2, \ldots, k \), the assumption implies

\[
2\ell|E| - (\ell - 1)n^2 \geq 2\ell \cdot \alpha n^2
\]

and therefore by (1) and since \( T_1 = n \)

\[
T_{\ell+1} \geq \frac{1}{\ell + 1} \frac{T_{\ell}}{n} 2\ell \alpha n^2 \geq \alpha nT_{\ell},
\]

which implies (8).

**Remark** Our result falls into the context of paragraph VI.1 of [2]. A well-known result by Turan ([3]) concerns the determination of the maximal number \( t_k(n) \) of edges in an \( n \)-graph such that \( T_{k+1} = 0 \).

The optimal graphs have the following structure:

For \( n = km + r, r < k \), partition \( \Omega_n \) into \( r \) sets with \( m + 1 \) vertices and \( k - r \) sets with \( m \) vertices and include exactly all edges connecting vertices of different sets.
Therefore one has for Turan’s function

\[ t_k(n) = \binom{r}{2}(m + 1)^2 + \binom{k-r}{2}m^2 + r(k-r)(m+1)m. \quad (9) \]

It is remarkable that our quite general inequality almost implies this identity. In fact, in an optimal graph clearly \( T_k \geq 1 \), because otherwise an edge could be added. Therefore from the inequality we conclude

\[ |\mathcal{E}| \leq \frac{n^2(k-1)}{2 \cdot k} \quad (10) \]

and if \( n \) is a multiple of \( k \), that is, \( n = m \cdot k \), then (10) takes the form \( |\mathcal{E}| \leq m^2 \binom{k}{2} \) and thus the bound in (9) follows.

For general \( n = km + r \) an easy calculation shows that the bound in (10) is tight, if \( \frac{(k-r)r}{2k} < 1 \). This is for instance always the case also for \( r = 1, 2 \).