On Source Coding with Side Information via a Multiple-Access Channel and Related Problems in Multi-User Information Theory

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Abstract—A simple proof of the coding theorem for the multiple-access channel (MAC) with arbitrarily correlated sources (DMCS) of Cover—El Gamal—Salehi, which includes the results of Ahlswede for the MAC and of Siegmund—Wolf for the DMCS and the MAC as special cases, is first given. A coding theorem is introduced and established for another type of source-channel matching problem, i.e., a system of source coding with side information via a MAC, which can be regarded as an extension of the Ahlswede—Körner—Wyner type noiseless coding system. This result is extended to a more general system with several principal sources and several side information sources subject to cross observation at the encoders in the sense of Han. The regions are shown to be optimal in special situations. Dueck's example shows that this is in general not the case for the result of Cover—El Gamal—Salehi and the present work. In another direction, the achievable rate region for the modulo-two sum source network found by Körner—Marton is improved. Finally, some ideas about a new approach to the source-channel matching problem in multi-user communication theory are presented. The basic concept is that of a correlated channel code. The approach leads to several new coding problems.

I. INTRODUCTION

It is well-known that Shannon's paper [1] was the starting point of multi-user information theory and it still seems, at least to us, that this paper is not given the attention that it deserves. In our judgment one of the most important problems raised is that of transmitting correlated messages over a noisy channel with two (or more) senders. This problem is still far from being completely understood. For the ease of our later reference and discussions we repeat here what Shannon wrote in the section Attainment of the Outer Bound with Dependent Sources [1, sect. 14, p. 636] (the numbers were inserted in the text by the authors for later reference):

With regard to the outer bound there is an interesting interpretation relating to a somewhat more general communication system. Suppose that the message sources at the two ends of our channel are not independent but statistically dependent. Thus, one might be sending weather information from Boston to New York and from New York to Boston. The weather at these cities is of course not statistically independent. 1) If the dependence were of just the right type for the channel or if the messages could be transformed so that this were the case, then it may be possible to attain transmission at the rates given by the outer bound. For example, in the multiplying channel just discussed, suppose that the messages at the two ends consist of streams of binary digits which occur with the dependent probabilities given by Table III. Successive $x_1$, $x_2$ pairs are assumed independent. Then by merely sending these streams into the channel (without processing) the outer bound curve is achieved at its midpoint.

It is not known whether this is possible in general. 2) Does there always exist a suitable pair of dependent sources that can be coded to give rates $R_1$, $R_2$ within $\epsilon$ of any point in the outer bound? 3) This is at least often possible in the noiseless memoryless case, that is, when $y_1$ and $y_2$ are strict functions of $x_1$ and $x_2$ (no channel noise). 4) The source pair defined by the assignment $p(x_1, x_2)$ that produces the point in question is often suitable in such a case without coding as in the above example.

Now, in Shannon’s notation, $x_i$ (resp. $y_i$) are the inputs (resp. outputs) at terminal $i$ ($i = 1, 2$) for the two-way channel (TWC) and the outer bound $G_0$ is the set of all pairs $(R_1, R_2)$ with

$$ R_1 = I(X_1; Y_2|X_2), \quad R_2 = I(X_2; Y_1|X_1), $$

(1.1)

where $X_1$, $X_2$ are dependent input variables and $Y_1$, $Y_2$ the output variables induced by the channel.

Shannon showed that the inner bound $G_*$, defined as convex hull of the set of rate pairs obtained in (1.1) for independent $X_1$, $X_2$, is an achievable rate region in the case of independent messages. Recently Dueck [2] showed that $G_*$ is in general not the capacity region $\mathcal{C}$ in the case of independent messages and $\mathcal{C}$ is still not known.

New progress in multi-user communication began by considering a simpler channel model, namely, that of a
multiple-access channel (MAC; also mentioned in [1]). Ahlswede [3] was the first to establish the capacity region of this channel in case of independent messages and subsequently Stepanian and Wolf [4] found the region of this channel for the following situation:

a) The correlated message statistic is described by a
   correlated memoryless source;

b) the MAC is noiseless.

They also found the region for an arbitrary MAC and a
certain special correlated message statistic in [5]. The result
of [4] can also be viewed as solution to sentence 3) men-
tioned in the quote. In case of a noiseless TWC formula
(1.1) gives

$$R_1 = H(X_1|X_2), \quad R_2 = H(X_1|X_1).$$

(1.2)

Notice that by the result of [4] sender 2, who knows $X_2$,
can be informed about $X_1$ with a rate $H(X_1|X_2)$, and
sender 1, who knows $X_1$, can be informed about $X_2$ with a
rate $H(X_1|X_1)$.

Considering again the MAC, which is better understood
than the TWC because the feedback problem is not pre-
sent. Cover, El Gamal, and Salehi [6] recently found a way
of using the dependency structure of the correlated mes-
sage source for the channel coding and thus obtained a
general coding theorem for the MAC which includes as
special cases the results of [3], [4], [5]. (The close connec-
tions between the results of [3] and [4] are explained for
instance in [7, part I, sect. 6]).

Dueck recently showed [8] that the approach of [6] does
not always give the full capacity region; it only does if the
dependency structure of the source fits nicely with the
dependency structure of the channel.

It seems that Shannon gave not only the direction but
also understood the situation quite well in the remarka-
ble sentence 3) of the quote. In the last Section VII of this
paper we will present some results and ideas, which we
hope will be helpful in making some further progress in the
direction indicated in sentence 1).

But the larger part of this paper (Sections II, III, and IV)
relates to the approach to the correlated source-multiple-
access channel matching problem given by Cover, El
Gamal, and Salehi [6]. They introduced an interesting
coding technique based on a kind of correlation-preserving
mapping.

In Section II we look at their coding theorem from the
viewpoint of cross observation at the encoders (Han [9],
[10]), revealing that the heart of their theorem consists in a
simpler but elegant version and that the theorem itself has
a simple proof.

In Section III we introduce another type of source-chan-
nel matching problem, i.e., a system of source coding with
side information via a multiple-access channel, which may
be regarded as an extension of the Ahlswede-Körner-
Wyner type noiseless source coding system ([11], [12]). For
this system we establish a matching condition.

In Section IV we consider a more general system with
several principal sources and several side information
sources subject to cross observation at the encoders, and
we establish a sufficient condition for the matching of this
system.

In Section V we present an achievable rate region for a
channel with side information at the decoder, which is
perhaps one of the simplest cases for which the converse is
presently not known.

In Section VI we describe a new achievable rate region
for the modulo-two sum source network considered by
Körner and Marton [13], which is a special but very
instructive case of the general two-helper source network
introduced in [24].

Even though most of our results are (except for special
cases) incomplete in the sense that no converses are proved,
we still feel that the results obtained, the problems
proposed, and the handy formalism provided will be of
some benefit for the advancement of the subject of multi-
user communication theory.

II. A New Look at and Simple Derivation of
the Cover - El Gamal - Salehi Coding Theorem

We establish first the fairly simple theorem 1 below and
then show that it contains the result of [6] as a special case.
Consider a memoryless multiple-access channel (MAC) $M$,
with input alphabets $X_1$, $X_2$ (finite), an output alphabet $Y$
(finite), and the transmission probabilities

$$w(y|x_1, x_2), \quad \text{for } y \in Y, \quad x_1 \in X_1, \quad x_2 \in X_2.$$  

Let $S = (S_1, S_2, S_3)$ be a multiple information source,
where $S_1$, $S_2$, $S_3$ are arbitrarily correlated random
variables with values in finite sets $S_1$, $S_2$, $S_3$, respectively.
Denote by $S_i^n$ an independent identically distributed (i.i.d.)
n-sequence of $S_i$ ($i = 1, 2, 3$).

Let us consider the following joint system of the source
$S$ and the channel $M_2$ with cross observation at the encoders (Fig. 1). The encoder $\phi_1$ observes a pair $(S_1^n, S_2^n)$ and
maps it to an input n-sequence $X_1^n$:

$$\phi_1: S_1^n \times S_2^n \rightarrow X_1^n.$$  

Similarly, the encoder $\phi_2$ observes $(S_2^n, S_3^n)$ and maps it to
an input n-sequence $X_2^n$:

$$\phi_2: S_2^n \times S_3^n \rightarrow X_2^n.$$  

The decoder $\psi$ observes an output n-sequence $Y^n$ and maps it to an element $(S_1^n, S_2^n, S_3^n)$:

$$\psi: Y^n \rightarrow S_1^n \times S_2^n \times S_3^n.$$  

The probability of error $P_e$ is given by

$$P_e = \Pr \left( S_2^n S_3^n S_1^n \neq S_2^n S_3^n S_1^n \right).$$

Definition 1: The source $S = (S_1, S_2, S_3)$ is said to be
admissible for the channel $M_2$ if for any $0 < \lambda < 1$ and
sufficiently large $n$ there exist encoding functions $\phi_1, \phi_2,
and a decoding function $\psi$ for which $P_e < \lambda$.

In order to obtain a sufficient condition for admissibil-
ity, it will be convenient to consider an associated test
channel as follows (cf. Han [9]).
Next, for the original channel $M_j$, define encoding functions $\phi_1, \phi_2$ and a decoding function $\psi$ by

$$
\phi_1(s_1^n, s_2^n) = f_1(\phi_1(s_1^n), \phi_2(s_2^n)),
$$

$$
\phi_2(s_2^n) = f_2(\phi_1(s_1^n), \phi_2(s_2^n)),
$$

$$
\psi = \psi^*.
$$

where $f_1(e, w) = f_1(v_1, w_1, \cdots, v_n, w_n)$ for $v = (v_1, \cdots, v_n)$, $w = (w_1, \cdots, w_n)$ ($i = 1, 2$). Clearly, these $\phi_1, \phi_2, \psi$ yield the same probability of error $\lambda$ for the channel $M_j$ too.

Q.E.D.

Now let us apply Theorem 1 to the following special case: let $S, T$ be arbitrarily correlated sources, let $K$ be the common variable of $S$ and $T$ in the sense of Gacs and Körner [14], and set $S_1 = S, S_2 = K, S_3 = T$. In this case our system (Fig. 1) is equivalent to the system considered by Cover–ElGamal–Salehi (Fig. 3). Thus, as a consequence of Theorem 1, we have the following result.

**Theorem 2:** (Cover–ElGamal–Salehi [6]) If there exist some $\tilde{Q}, X_1, X_2, Y$ such that

$$
\Pr\{S = s, T = t, \tilde{Q} = \tilde{q}, X_1 = x_1, X_2 = x_2, Y = y\} = p(s, t, \tilde{q}) p(x_1|s, \tilde{q}) p(x_2|t, \tilde{q}) w(y|x_1, x_2)
$$

and such that

$$
H(S|T) < I(X_1; Y|X_2\tilde{Q}),
$$

$$
H(T|S) < I(X_1; X_2|\tilde{Q}),
$$

$$
H(ST|K) < I(X_1; X_2|\tilde{Q}),
$$

$$
H(ST) < I(X_1; X_2),
$$

then the source $(S, T)$ is admissible for the channel $M_2$.

**Proof:** With the choice $U_2 = \tilde{Q}$ condition (2.2) is in the present situation equivalent with the following seven inequalities:

$$
H(S|T) < I(U_1; Y|U_2T\tilde{Q}),
$$

$$
H(T|S) < I(U_1; Y|U_2\tilde{Q}),
$$

$$
H(K|ST) < I(\tilde{Q}; Y|U_2ST),
$$

$$
H(SK|T) < I(U_1\tilde{Q}; Y|U_2T),
$$

$$
H(TK|S) < I(U_1\tilde{Q}; Y|U_2S),
$$

$$
H(ST|K) < I(U_1U_2; Y|K\tilde{Q}),
$$

$$
H(STK) < I(U_1U_2\tilde{Q}; Y).
$$

It is easy to check that (2.8) and (2.9) imply (2.11) and (2.12), respectively, and (2.10) is trivial (cf. Remark 1).
On the other hand, the right-hand side of (2.8) can be rewritten using the assumed Markov chain properties and (2.1) as

\[
I(U_j; Y| U_j TKQ) = I(U_j; X_j; Y| U_j TQ) \\
= I(X_j; Y| U_j TQ) \\
= I(X_j; Y| U_j X_j TQ) \\
= I(X_j; Y| X_j TQ).
\]

Similarly, for (2.9), (2.13), and (2.14) we have

\[
I(U_j; Y| U_j SKQ) = I(X_j; Y| X_j S_j Q), \\
I(U_j, U_j; Y| KQ) = I(X_j X_j; Y| KQ), \\
I(U_j U_j; Y) = I(X_j X_j; Y).
\]

Q.E.D.

Remark 2: The way of deriving the theorem from Theorem 1 reveals that the heart of it consists in its special but elegant case with \( \hat{Q} \) being a constant:

\[
H(S|T) < I(X_1; Y| X_2 T),
\]

(2.15)

\[
H(T|S) < I(X_2; Y| X_1 S),
\]

(2.16)

\[
H(S T) < I(X_1 X_2; Y).
\]

(2.17)

Note here that (2.7) implies (2.6) if \( \hat{Q} \) is a constant.

III. Source Coding with Side Information via a Multiple-Access Channel

The system considered by Cover–El Gamal–Soltani may be regarded as an extension of the Slepian–Wolf type noiseless source coding system [4]. In this section we consider a parallel extension of the Ahlswede–Körner–Wyner type noiseless source coding system with side information [11], [12].

Let \( M_2 \) be the multiple-access channel as specified in Section II, and let \( S = (S, T) \) be an arbitrarily correlated source with alphabets \( S, T \) (finite sets), respectively. We shall consider here the following joint system of \( S \) and \( M_2 \) as depicted in Fig. 4. The encoders \( \phi_1, \phi_2 \) are defined by:

\[
\phi_1: S^n \rightarrow \mathcal{X}_1^n, \\
\phi_2: S^n \rightarrow \mathcal{X}_2^n.
\]

The decoder \( \psi \) observes an output \( n \)-sequence \( Y^n \) and maps it to an element \( \hat{S}^n \) of \( S^n \). \( \psi: Y^n \rightarrow \hat{S}^n \). Since in this system the purpose of the decoder \( \psi \) is to reliably reproduce the source \( S^n \) alone, the probability of error \( P_e(S) \) is defined by

\[
P_e(S) = \Pr(\hat{S}^n \neq S^n).
\]

Definition 2: The source \( S = (S, T) \) is said to be \( \beta \)-admissible for the channel \( M_2 \) if for any \( 0 < \lambda < 1 \) and sufficiently large \( n \) there exist encoding functions \( \phi_1, \phi_2 \) and a decoding function \( \psi \) such that \( P_e(S) < \lambda \).

Let \( Q, V \) be any random variables with values in finite sets \( \mathcal{X}_1, \mathcal{X}_2 \), respectively, such that

\[
\Pr(S = s, T = t, Q = q, V = v) = p(s, t) p(q) p(x_1, x_2) p(v|t, q)
\]

\[
= p(x_1, t) p(q) p(x_1, x_2) w(y|x_1, x_2),
\]

(3.1)

where \( x_1, x_2 \) take values in the input alphabets \( \mathcal{X}_1, \mathcal{X}_2 \), respectively, and \( Y \) in the output alphabet \( \mathcal{X}_1 \) \( (Q \) is the time-sharing parameter).

Then we have Theorem 3:

Theorem 3: If there exist some \( Q, V, X_1, X_2, Y \) satisfying (3.1) for which

\[
H(S|VQ) < I(SX_1; Y|VQ),
\]

(3.2)

\[
H(S|VQ) + I(T; V|Q) < I(SX_1 V; Y|Q).
\]

(3.3)

then the source \( (S, T) \) is \( S \)-admissible for the channel \( M_2 \). Here it is sufficient to consider only \( Q, V \) such that the cardinalities \( ||Q||, ||V|| \) of the ranges of \( Q, V \) are bounded by

\[
||V|| \leq ||S|| + 3, \quad ||Q|| \leq 4.
\]

Remark 3: Conditions (3.2), (3.3) are equivalent to the following seemingly stronger conditions:

\[
H(S|VQ) < I(SX_1; Y|VQ),
\]

(3.4)

\[
I(T; V|SQ) < I(V, Y|SX_1 Q).
\]

(3.5)

\[
H(S|VQ) + I(T; V|Q) < I(SX_1 V; Y|Q).
\]

(3.6)

In fact, suppose that \( Q, V, X_1, X_2, Y \) satisfy (3.2), (3.3). If \( I(T; V|SQ) < I(V, Y|SX_1 Q) \) for those variables, then (3.4)–(3.6) follows. On the other hand, if \( I(T; V|SQ) \geq I(V, Y|SX_1 Q) \), by rewriting (3.3) we have

\[
H(S) < I(SX_1; Y|Q) + I(V, Y|SX_1 Q) - I(T; V|SQ)
\]

\[
\leq I(SX_1; Y|Q).
\]

This coincides with (3.4)–(3.6) with \( V \) set constant (cf. Remark 1).

Proof of Theorem 3: In view of Remark 3, it suffices to prove the admissibility under conditions (3.4)–(3.6). In proving the theorem we use the fundamental properties of jointly typical sequences (cf. Berger [15], Han and Kobayashi [16]. The notion used is slightly different from the one of Wolfowitz [17]). The set of all \( \epsilon \)-typical sequences for a random variable \( Z \) is denoted by \( T_\epsilon(Z) \), and, for \( \epsilon \in (0, 1) \),

\[
T_\epsilon(Z|W) = \{ z|z\in T_\epsilon(ZW) \}.
\]
1) Auxiliary Code: First generate one random n-sequence \( q = (q_1, \ldots, q_n) \) according to \( \prod P(q_i) \).

Next, fix any \( R \) such that \( R > I(T; V^{2}) \) and take \( L = \exp[nR] \) mutually independent n-sequences \( \tilde{V}_1, \ldots, \tilde{V}_L \) with values taken equiprobably in \( T_r(V^2|q) \), and set \( \tilde{V} = (\tilde{V}_1, \ldots, \tilde{V}_L) \).

Note here that \( \tilde{V}_1, \ldots, \tilde{V}_L \) depend on the value of \( q \). Since (3.1) implies that \( S, T, V \) (given \( Q \)), form a Markov chain in this order, there exists for sufficiently large \( n \) a function \( V^* = g(T^n; \tilde{V}_1, \ldots, \tilde{V}_L) \) such that \( V^* = \tilde{V} \) for \( i = 1, \ldots, L \) and

\[
\Pr \left( (S^n, T^n, V^*) \in T_r(STV|q) \right) \geq 1 - \delta, \tag{3.7}
\]

where \( \delta = \delta(\epsilon) \to 0 \) as \( \epsilon \to 0 \) (a conditional version of Lemma 3.3 of Han and Kobayashi [16]).

2) Random Code Generation: For each \( s = (s_1, \ldots, s_n) \in S^n \) generate one random n-sequence \( x_i(s) = (x_{1i}, \ldots, x_{ni}) \in \mathfrak{X}_i \) according to

\[
\prod_{k=1}^{n} p(x_{ik}|s_k, q_k).
\]

For each \( t = (t_1, \ldots, t_n) \in \mathfrak{Y}^n \) generate one random n-sequence \( x_i(t, \mathfrak{v}^*) = (x_{1i}, \ldots, x_{ni}) \in \mathfrak{X}_i \) according to

\[
\prod_{k=1}^{n} p(x_{ik}|t_k, \mathfrak{v}^*_k, q_k).
\]

where \( \mathfrak{v}^* = (\mathfrak{v}_1^*, \ldots, \mathfrak{v}_n^*) = g(t; \tilde{V}_1, \ldots, \tilde{V}_L) \).

3) Encoding: Define the encoding functions \( \phi_i: S^n \rightarrow \mathfrak{X}_i, \phi_i: S^n \rightarrow \mathfrak{X}_i \) by

\[
\phi_i(s) = x_i(s), \tag{3.8}
\]

\[
\phi_i(t) = x_2(t, \mathfrak{v}^*) = x_2(t, g(t; \tilde{V}_1, \ldots, \tilde{V}_L)). \tag{3.9}
\]

4) Decoding: Let \( X_1^n = x_1(S^n), X_2^n = x_2(T^n, V^*), \) and indicate by \( Y^n \) the output n-sequence induced on \( \mathfrak{Y}^n \) from \( X_1^n, X_2^n \) via the channel \( M_2 \). If \( (s, \mathfrak{v}^*) \) is the only element of \( S^n \times \mathfrak{Y} \) such that

\[
(s, x_1(s), \mathfrak{v}^*, Y^n) \in T_r(SX|V^2|q). \tag{3.10}
\]

Then define the decoding function \( \psi: \mathfrak{Y}^n \rightarrow S^n \) by \( \psi(Y^n) = s \), otherwise let \( \psi(Y^n) \) be arbitrary (the decoder is to be informed about the value of \( q \)).

5) Probability of Error: Denoting by \( E(s, \mathfrak{v}^*) \) the event (3.10) for \( (s, \mathfrak{v}^*) \in S^n \times \mathfrak{Y} \), we have the following expression for the probability of error:

\[
P_e = \Pr \left\{ E^c(S^n, V^*) \cup \bigcup_{\mathfrak{v}^* \in S} E(s, \mathfrak{v}^*) \right\}
\]

\[
\leq \Pr \left( (S^n, T^n, V^*) \in T_r(STV|q) \right) + \Pr \left( E^c(S^n, V^*)|(S^n, T^n, V^*) \in T_r(STV|q) \right)
\]

\[
+ \Ex \left( \sum_{\mathfrak{v}^* \in S} \Pr \left( E(s, \mathfrak{v}^*|F_0(S^n, T^n, V^*)) \right) \right), \tag{3.11}
\]

where "c" indicates the complement; \( \Ex(...) \) denotes expectation, and \( F_0(S^n, T^n, V^*) \) denotes the event

\[
((S^n, T^n, V^*) \in T_r(STV|q)) \cap E(S^n, V^*).
\]

Note that the range of \( \mathfrak{v}^* \) in the above sums is restricted to be within \( \mathfrak{Y} \).

From (3.7) we have

\[
\Pr \left( (S^n, T^n, V^*) \in T_r(STV|q) \right) \leq \delta. \tag{3.12}
\]

Next, by the Markov properties among \( S, V, X_1, X_2, Y \) given \( Q \) (derived from (3.1)) and from the way of generating the random sequences \( x_i(s), x_2(t, \mathfrak{v}^*) \), it follows that

\[
\Pr \left( E^c(S^n, V^*)|(S^n, T^n, V^*) \in T_r(STV|q) \right) \leq \epsilon. \tag{3.13}
\]

On the other hand, the last term in (3.11) can be decomposed as follows:

\[
\Ex \left( \sum_{\mathfrak{v}^* \in S} \Pr \left( E(s, \mathfrak{v}^*|F_0(S^n, T^n, V^*)) \right) \right)
\]

\[
= \Ex \left( \sum_{\mathfrak{v}^* \in S} \Pr \left( E(s, \mathfrak{v}^*|F_0(S^n, T^n, V^*)) \right) \right)
\]

\[
+ \Ex \left( \sum_{\mathfrak{v}^* \in S} \Pr \left( E(s, \mathfrak{v}^*|E_0(S^n, T^n, V^*)) \right) \right)
\]

\[
+ \Ex \left( \sum_{\mathfrak{v}^* \in S} \Pr \left( E(s, \mathfrak{v}^*|E_0(S^n, T^n, V^*)) \right) \right). \tag{3.14}
\]

Denote the first, second, and third terms on the right-hand side of (3.14) by \( P_1, P_2, P_3 \), respectively.

a) Evaluation of \( P_1 \): Set

\[
\mathfrak{d}_1 = \{ s \in S^n, s \in T_r(S^nX^nY^n|q) \},
\]

and for each \( s \in \mathfrak{d}_1 \),

\[
\mathfrak{B}_1(s) = T_r(X_1|S^nX^nY^n|q),
\]

\[
q_1(s) = \max_{x_1 \in \mathfrak{B}_1(s)} \Pr \left( x_1(s) = x_1 \right)
\]

\[
\leq \exp \left[ -n(H(X_i|S^nQ) - 2\epsilon) \right].
\]

Then, for any \( s \in \mathfrak{d}_1 \),

\[
P_1(s) = \Pr \left( E(s, \mathfrak{v}^*|F_0(S^n, T^n, V^*)) \right)
\]

\[
\leq \Ex \left( \max_{s \in \mathfrak{d}_1} (|X_1(s)| \cdot q_1(s)) \right)
\]

\[
\leq \exp \left[ n(H(X_i|S^nQ) + 2\epsilon) \right]
\]

\[
\cdot \exp \left[ -n(H(X_i|S^nQ) - 2\epsilon) \right]
\]

\[
= \exp \left[ -n(I(X_i; Y|S^nQ) - 4\epsilon) \right]
\]

\[
= \exp \left[ -n(I(SX_i; Y|Q) - I(S; Y|Q) - 4\epsilon) \right]. \tag{3.15}
\]

On the other hand,

\[
|\mathfrak{d}_1| \leq \exp \left[ n(H(S|YQ) + 2\epsilon) \right]
\]

\[
= \exp \left[ n(H(S|YQ) - I(S; Y|Q) + 2\epsilon) \right]. \tag{3.16}
\]
From (3.15) and (3.16),

\[ P_1 = \text{Ex} \left( \sum_{s \in \mathcal{D}_1} P_1(s) \right) \leq \text{Ex} \left( \left[ \mathbb{E}_{l_1} \cdot \max_{s \in \mathcal{D}_1} P_1(s) \right] \right) \]

\[ \leq \exp \left( -n \left( I(SX_t; Y|V) - H(S|VQ) - 6\epsilon \right) \right). \]

Consequently, condition (3.4) yields

\[ P_1 < \epsilon \] (3.17)

for sufficiently large \( n \).

b) Evaluation of \( P_2 \): Noting that \( \tilde{V}_1, \cdots, \tilde{V}_L (\neq V^*) \) are all uniformly distributed on \( T_\epsilon(V|Q) \), we have

\[ P_2 = \text{Ex} \left( \sum_{s \in \mathcal{D}_2} \text{Pr} \left( (E(S^n, v^*), F_0(S^n, T^n, v^n) \right) \right) \]

\[ \leq \left( L - 1 \right) \exp \left[ \frac{n \left( H(V|SV) + 2\epsilon \right)}{1 - \epsilon} \right] \exp \left[ \frac{n \left( H(V|Q) - 2\epsilon \right)}{1 - \epsilon} \right] \]

\[ \leq \left( L - 1 \right) \exp \left[ -n \left( I(V; Y|Q) - 5\epsilon \right) \right] \] (3.18)

for sufficiently large \( n \). Since \( R (L = \exp (nR')) \) can be arbitrarily chosen so long as \( R > I(T; V|Q) \), we may set

\[ R = I(T; V|Q) + \epsilon. \]

Therefore,

\[ P_2 < \exp \left[ -n \left( I(V; SXY|Q) - I(T; V|Q) - 6\epsilon \right) \right] \]

\[ = \exp \left[ -n \left( I(V; Y|Q) - I(T; V|Q) - 6\epsilon \right) \right]. \] (3.19)

which implies by condition (3.5) that

\[ P_2 < \epsilon \] (3.20)

for sufficiently large \( n \).

c) Evaluation of \( P_3 \): Set

\[ \mathcal{D}_3 = \left\{ (s, v^*) | s = S^*, v^* = V^*, (s, v^*) \in T_\epsilon(SV|Y|Q) \right\}, \]

and for each \( (s, v^*) \in \mathcal{D}_3, \)

\[ \mathcal{B}_3(s, v^*) = T_\epsilon(X|sv^*Y|Q), \]

\[ q_3(s, v^*) = \max_{x_1, v_1 \in \mathcal{B}_3(s, v^*)} \text{Pr} \left( x_1(s) = x_1 \right). \]

Then for every \( (s, v^*) \in \mathcal{D}_3, \)

\[ P_3(s, v^*) = \text{Pr} \left( E(s, v^*)|E(S^n, V^n) \right) \]

\[ \leq \text{Ex} \left( \max_{s \in \mathcal{D}_3} \left( \mathcal{B}_3(s, v^*) \right) \cdot q_3(s, v^*) \right) \]

\[ \leq \exp \left[ n \left( H(X|SVYQ) + 2\epsilon \right) \right] \exp \left[ -n \left( H(X|SVQ) - 2\epsilon \right) \right] \]

\[ = \exp \left[ -n \left( I(X_1; Y|SVQ) - 4\epsilon \right) \right]. \]

Therefore,

\[ P_3 = \text{Ex} \left( \sum_{v^* \in \mathcal{D}_3} P_3(s, v^*) \right) \]

\[ = \text{Ex} \left( \mathcal{D}_3 \right) \exp \left[ -n \left( I(X_1; Y|SVQ) - 4\epsilon \right) \right] \]

\[ = \text{Ex} \left( \mathcal{D}_3 \right) \exp \left[ -n \left( I(SX_1Y; Y|Q) \right) - I(SV; Y|Q) - 4\epsilon \right]. \] (3.21)

On the other hand, again from the uniform distribution property of \( \tilde{V}_1, \cdots, \tilde{V}_L (\neq V^*) \) on

\[ T_\epsilon(V|Q) \]

we have

\[ \text{Ex} \left( |\mathcal{D}_3| \right) \]

\[ \leq \frac{(L - 1) \exp \left[ n \left( H(V|YQ) + 2\epsilon \right) \right]}{(1 - \epsilon) \exp \left[ n \left( H(V|Q) - 2\epsilon \right) \right]} \]

\[ - \exp \left[ n \left( H(S|YVQ) + 2\epsilon \right) \right] \]

\[ \leq (L - 1) \exp \left[ n \left( H(S|YQ) - I(SV; Y|Q) + 4\epsilon \right) \right] \]

\[ \leq \exp \left[ n \left( I(T; V|Q) + H(S|VQ) - I(SV; Y|Q) + 8\epsilon \right) \right]. \] (3.22)

From (3.21) and (3.22),

\[ P_3 < \exp \left[ -n \left( I(SXY; Y) - H(S|VQ) - I(T; V|Q) - 12\epsilon \right) \right]. \]

Hence, condition (3.6) yields

\[ P_3 < \epsilon \] (3.23)

for sufficiently large \( n \).

Summarizing (3.11)–(3.14), (3.17), (3.20), (3.23), we can conclude that \( P_2 < \delta + 4\epsilon \), Q.E.D.

We present now several special cases of Theorem 3.

Corollary 1: If there exist \( X_1, X_2, Y \) such that

\[ \text{Pr} \left( S = s, T = t, X_1 = x_1, X_2 = x_2, Y = y \right) = p(s, t) p(x_1|s) p(x_2|t) w(y|x_1, x_2) \] (3.24)

then the source \( (S, T) \) is \( S \)-admissible.

Proof: Let \( Q = V = \phi \) (\( \phi \) is a constant variable) in (3.2), (3.3) of Theorem 3 (also, cf. Remark 1). Q.E.D.

Remark 4: The right-hand side of (3.25) may be interpreted as follows. First, decompose \( I(SX_1; Y) \) as

\[ I(SX_1; Y) = I(S_1; Y) + I(S; Y|X_1). \]

The first term \( I(S_1; Y) \) represents the information passing directly from the input \( X_1 \) to the output \( Y \) when we use the given multiple-access channel \( M_2 \) as a single-user channel with the "random state \( T \)" correlated to the source \( S \); the second term \( I(S; Y|X_1) \) represents the information passing from the input \( S \) via the intermediate terminal \( T \) to the output \( Y \), without passing through \( X_1 \).

Here, in order to establish some application of Corollary 1, let us consider a single-user channel with "random state \( T \)" as follows. The transition probabilities are specified by

\[ w(y|x, r), \quad x \in \mathcal{X}, y \in \mathcal{Y}, r \in \mathcal{S}, \]

where \( \mathcal{X}, \mathcal{Y}, \mathcal{S} \) are the input and output alphabets, respec-
tively, and \( t \) indicates a value of random state distributed according to the random variable \( T \). We are required to reliably send a source \( S \) (taking values in \( \mathbb{X} \)) which is correlated to \( T \). The encoder \( \phi : \mathbb{S} \rightarrow \mathbb{X}^n \) maps \( \mathbb{S}^* = (S_1, \cdots, S_n) \) to the input \( n \)-sequence \( X^n \), and the decoder \( \psi : \mathbb{X}^n \rightarrow \mathbb{S}^* \) maps the output \( n \)-sequence \( Y^n \) to an estimate \( \hat{S}^* \) of \( S^* \). The random state \( T \), for the \( i \)-th channel use is correlated only to the \( i \)-th source output \( S_i \), and the joint distribution of \((S, T)\) is i.i.d. of the generic random variable \((S, T)\). We designate this kind of channel by \( C(T) \).

**Theorem 4 (Coding Theorem for \( C(T) \)):** If there exists \( X, Y \) such that
\[
\Pr(S = s, T = t, X = x, Y = y) = p(s, t)p(x|s)w(y|x, t)
\]
and
\[
H(S) < I(SX; Y),
\]
then the source \( S \) is admissible for the channel \( C(T) \) with correlated random state \( T \).

Conversely, if the source \( S \) is admissible for \( C(T) \), then (2.27) with \( \leq \) replacing \( < \) holds for some \( X, Y \) satisfying (2.26).

**Proof:** See Appendix II.

**Remark 5:** Theorem 4 establishes a necessary and sufficient condition for the admissibility of the source for the channel with random state, but with no statement on the admissibility for a trivial situation \( H(S) = I(SX; Y|Q) \). This situation should be further examined not in a general but depending on specific characteristics of each particular channel.

Corollary 1 treats a case where the encoder \( \phi_2 \) for the side source \( T^n \) no longer carries out block encoding. On the other hand, the following example demonstrates another case where the encoder \( \phi_1 \) for the principal source \( S^* \) attains no block encoding, but componentwise random encoding. (That is, \( \phi_1 : \mathbb{S} \rightarrow \mathbb{X}^n \) is called a componentwise random encoder if there exist independent random functions \( h_1, \cdots, h_n \) from \( \mathbb{S} \) to \( \mathbb{X} \) such that \( \phi_1(t) = (h_1(t), \cdots, h_n(t)) \), where \( t = (t_1, \cdots, t_n) \).) In this latter case, the auxiliary variable \( V \) actually intervenes and hence block encoding is essentially needed for the source \( T^n \).

To show an example, let us consider a special multiple-access channel \( M_2^0 \) with some deterministic function \( f \): \( \mathbb{S} \rightarrow \mathbb{X} \) such that \( w(y|x_1, x_2) = 0 \) for \( x_2 \neq f(y) \). In other words, the channel \( M_2^0 \) is such that one of the inputs \( x_2 \) is noiselessly transmitted to the receiver.

**Theorem 5 (Coding Theorem for \( M_2^0 \)):** If there exist some \( Q, V, X_1, X_2 \) such that
\[
\Pr(S = s, T = t, Q = q, X_1 = x_1, X_2 = x_2, Y = y) = p(t, s)p(q)p(v|t, q)p(x_1|s, q)w(y|x_1, x_2)
\]
for which
\[
H(S|V < I(X_1; Y|X_2, V|Q),
\]
\[
H(S|V > I(T; V|Q) < I(X_1; Y|Q),
\]
then the source \((S, T)\) is \( S \)-admissible for \( M_2^0 \) by a componentwise random encoder \( \phi_1 \). Conversely, if the source \((S, T)\) is \( S \)-admissible for \( M_2^0 \) with a componentwise random encoder \( \phi_1 \), then conditions (3.29), (3.30) with "\( < \)" replaced by "\( \leq \)" have to be satisfied for some \( Q, V, X_1, X_2 \) satisfying (3.28). Here it is sufficient to consider only \( V, Q \) such that
\[
\|V\| \leq |\mathbb{S}| \cdot |\mathbb{X}_1| + 3, \quad \|Q\| \leq 4.
\]

**Proof:** See Appendix II.

**Example 1:** Consider the case where \( M_2^0 \) is a pair of noiseless channels, i.e., \( Y = X_1X_2 \). If we put in (3.28) of Theorem 5 \( \phi = \phi_1, p(x_1|s, q) = p(x_1), p(x_2|v, q) = p(x_2) \), then conditions (3.29), (3.30) are reduced to
\[
H(S|V) < R_1,
\]
\[
H(S|V) + I(T; V) < R_1 + R_2,
\]
where \( R_1 = H(X_1), R_1 = H(X_2) \). Clearly, these conditions are implied by the conditions \( H(S|V) < R_1, I(T; V) < R_2 \) established by Ahlswede-Körner-Wyner [11], [12]. Therefore, Theorem 5 may be regarded as an extension of the noiseless source coding theorem with side information.

**Example 2:** Consider a case where \( S, T \) are independent. If in Theorem 3 we put \( p(x_1|s, q) = p(x_1), p(x_2|v, q) = p(x_2), \) replace \( V \) by \( VX_1 \), and then set \( V = \phi \), conditions (3.2), (3.3) reduce to \( H(S) < I(X_1; Y|X_2) \), and can therefore be replaced by
\[
H(S) < \max_{x_1, x_2} I(X_1; Y|X_2). \tag{3.31}
\]

**Example 3:** In the case \( S = T \) (total cooperation), Theorem 3 does not cover the optimal condition
\[
H(S) < \max_{p(x_1, x_2)} I(X_1, X_2; Y). \tag{3.32}
\]
This optimal condition is covered by Corollary 2 to appear in Section IV.

We conclude this section by giving a limiting expression of the condition for the admissibility. For any integer \( m = 1, 2, \cdots \), let \( \psi_1 : \mathbb{S} \rightarrow \mathbb{X}^m, \psi_2 : \mathbb{S} \rightarrow \mathbb{X}^m \) be any random function such that \( X^n = \phi_1(S_1^n), S_1^n, T^n, \psi_2(T_1^n) = X^n_2 \) form a Markov chain in this order, and indicate by \( Y^n \) the corresponding output variable on \( \psi_2 \).

**Theorem 6:** If
\[
H(S) < \sup_{m, \psi_1, \psi_2} \frac{1}{m} I(S^nX^n_1; Y^n),
\]
then the source \((S, T)\) is \( S \)-admissible for \( M_2^0 \). Conversely, if the source \((S, T)\) is \( S \)-admissible for \( M_2^0 \), then
\[
H(S) < \sup_{m, \psi_1, \psi_2} \frac{1}{m} I(S^nX^n_1; Y^n).
\]
Proof: The former part immediately follows from Corollary 1 by considering $S^m, T^n$ as "supersources" of length $m$ instead of $S, T$. The latter part is derived as follows: put
\[ r(\lambda) = n \lambda \log |S| + h(\lambda), \]
\[ \times (h(\lambda) = - \lambda \log (1 - \lambda) \log (1 - \lambda)), \]
then by Fano's inequality,
\[ m H(S) - r(\lambda) \leq H(S^n) - H(S^n|Y^n), \]
where $\lambda$ is the probability of error. $m$ is the block length of the code, and $X^n, Y^n$ are defined with respect to the encoders $\phi, \psi$ under consideration. Note that $r(\lambda)/m \rightarrow 0$ as $\lambda \rightarrow 0$.

Q.E.D.

Remark 6: Of course the "naive" characterization given in Theorem 6 cannot be used even in principal for numerical evaluations. However, since the so called single letter characterization involving auxiliary variables is often hard to get in multi-user theory, one should try as an alternate approach to get estimates on the speed of convergence in the "naive" approach. This may be a hard task, but there are no obvious reasons why this should be impossible.

IV. RESULTS FOR A MORE GENERAL SYSTEM

In the field of multiterminal nonacode source coding theory various kinds of source coding systems have been devised and studied. After Slepian-Wolf [4] treated the most basic and simplest case, a new dimension was added by Ahlswede-Körner [11] and Wyner [12] (and earlier, but in a weaker form, by Gray-Wyner [18]) by considering not only principal sources but also side information sources. In channel coding a certain kind of side information had been studied much earlier already by Shannon [21]. Various extensions of those results have been found and presently the most general ones seem to be those of Csiszár-Körner [19] and Han-Kobayashi [16]. A new direction in source coding was recently provided by Ahlswede [7], where multisources are described by specifying conditional distributions rather than joint distributions, that is, less knowledge about the source is assumed.

Here we stick to classical correlated source systems with several principal sources and several side information sources (helpers) as components whose coded versions are to be transmitted over a multiple-access channel with a "single" receiver.

Let $M_r$ be a multiple-access channel with $r$ input alphabets $\mathcal{X}_1, \ldots, \mathcal{X}_r$ (finite), an output alphabet $\mathcal{Y}$ (finite), and the transition probability
\[ p(y|x_1, \ldots, x_r), \quad y \in \mathcal{Y}, x_1 \in \mathcal{X}_1, \ldots, x_r \in \mathcal{X}_r, \]
\[ (4.1) \]
and define the input variables $X_j$ on $\mathcal{X}_j$, of the channel $M_r$ by $X_j = f_j(U_{\mathcal{S}_j})$ ($j = 1, \ldots, r$). Denote by $Y$ the output variable with values in $\mathcal{Y}$ induced form the $X_j$ via the channel $M_r$. The relation among $S, V_j, X_j, Y$ is illustrated in Fig. 5 (the time-sharing parameter).

Theorem 7: If there exist some $Q$: $V_j, \ldots, V_j, U_r, \ldots, U_r, f_1, \ldots, f_r$: $X_1, \ldots, X_r, Y$ as above for which for all $A$ with $A \in \Sigma$, $A \cap \Sigma^1 = \emptyset$,
\[ I(S^m; V^m|Q) < I(W_a; Y) + Q), \]
where we have put $W_a = S^mU_a$, for $j \in \Sigma^1$, $W_a = V^m$ for $j \in \Sigma^2$, then the source $S^m$ is $(S_1, \ldots, S_r)$-admissible for the channel $M_r$.

Remark 7: In (4.5) we may omit the corresponding condition if its left-hand side vanishes.
Proof: Both the test channel argument in Section II and the argument used in the proof of Theorem 2 can be extended to the general case under consideration (cf. Han [9]), from which condition (4.5) follows. Q.E.D.

We shall next give a limiting expression for the condition of the admissibility for this general system. Consider any integer \( m = 1, 2, \cdots \), and let \( \phi_i: \mathbb{S}_m^n \to \mathbb{Q}_m^n \) be any random functions \( (i \in \Sigma) \) such that

\[
\Pr(S^n = s, U^n = u, i \in \Sigma) = \rho(s_i, i \in \Sigma) \prod_{i \in \Sigma} \rho(u_i | s_i),
\]

where \( U^n = \phi(S^n) \). Let \( f_j: \mathbb{Q}_m^n \to \mathbb{X}_j^n \) be any deterministic functions \( (j = 1, \cdots, r) \) and set \( \mathbb{X}_j^n = f_j(U^n) \). Denote by \( \mathbb{Y}^n \) the output variable on \( \mathbb{Q}_m^n \) induced from \( \mathbb{X}_j^n \). Indicate by \( (a_j) \) the vector with components \( a_j \) indexed by \( A \subseteq \Sigma^{(1)} \) \( (A \neq \Phi) \) and define

\[
\mathbb{C}_m = \left\{ (a_j) | \sum_{j=1}^r a_j \leq 1/m I(S_n^n U^n; Y^n | S_n^n U^n) \text{ for some } f_j \right\},
\]

where \( A^v \) indicates the complement of \( A \) in \( \Sigma^{(1)} \); and

\[
\mathbb{C} = \bigcup_{m=1}^\infty \mathbb{C}_m.
\]

Note that \( \mathbb{C} \) is a bounded set because \( \mathbb{C}_m \) \( (m = 1, 2, \cdots) \) are all within bounded set. Denote by \( \overline{\mathbb{C}} \) the closure of \( \mathbb{C} \).

Theorem 8 (Limiting Expression): Let \( h_A = H(S_A ; S_A) \) for \( A \) with \( \phi = A \subseteq \Sigma^{(1)} \). If \( (h_A) \) is an internal point of \( \overline{\mathbb{C}} \), then the source \( (S_1, \cdots, S_r, S_{r+1}, \cdots, S_p) \) is \( (S_1, \cdots, S_r) \)-admissible. Conversely, if the source \( (S_1, \cdots, S_r, S_{r+1}, \cdots, S_p) \) is \( (S_1, \cdots, S_r) \)-admissible then \( (h_A) \in \overline{\mathbb{C}} \).

Proof: The former part is an immediate consequence of Theorem 7: Let \( \mathbb{V}_1 = \mathbb{g} (i \in \Sigma^{(2)}) \) and apply to the super sources \( \mathbb{S}_m^n \) instead of \( S_n \). The latter part follows analogously to the proof of Theorem 6. Q.E.D.

We are now in the position to prove a stronger version of Theorem 3. Let \( S_1 = S, S_2 = K, S_3 = T \) \( (K \) is the common variable of \( S \) and \( T \) in the sense of Gacs and Körner) and apply Theorem 7 to the special system depicted in Fig. 1 with \( (S_1^n, S_2^n, S_3^n) \) replaced by \( (S_1^n, S_2^n) \), where \( S_1, S_2 \) are principal, \( S_3 \) is a helper \( (\Sigma^{(1)} = \{1, 2\}, p = 3, a = 2, r = 2) \). Then we have the following result for the equivalent side information system depicted in Fig. 4.

**Theorem 9:** If there exists some \( Q, \dot{Q}, V, X_1, X_2, Y \) such that

\[
\Pr(S = s, T = t, K = k, Q = q, \dot{Q} = \dot{q}, X_1 = x_1, X_2 = x_2, Y = y) = p(s, t) p(q | \dot{q}) p(k | \dot{q}, q) p(x_1 | \dot{q}, q) p(x_2 | \dot{q}, q) w(y | x_1, x_2)
\]

for which

\[
H(S | KVQ) < I(SX_1; Y | KV \dot{Q}) \tag{4.6}
\]

then the source \( (S, T) \) is \( S \)-admissible for the channel \( M_2 \). Here it is sufficient to consider only \( V, Q, \dot{Q} \) such that

\[
||V|| \leq ||T|| - ||X_2|| + 7, \quad ||\dot{Q}|| \leq 8, \quad ||\dot{Q}|| \leq ||X_1|| + ||X_2|| + ||K|| + 4.
\]

Remark 8:

1) It is easy to see that Theorem 3 is a special case of Theorem 9. In fact, if we set \( \dot{Q} = \phi \) (a constant) in (4.7)–(4.10) then (4.8), (4.10) imply (4.7), (4.9), respectively, yielding Theorem 3.

2) The variable \( \dot{Q} \) seems somewhat similar to the time-sharing parameter \( Q \), but is different in that \( \dot{Q} \) is used here also to carry the common information \( K \) as is the case in the theorem of Cover-El Gamal-Salehi.

**Corollary 2:** Suppose that there exists a deterministic function \( T = f(S) \), i.e., \( K = T \); then the following holds. If there exist \( \dot{Q}, X_1, X_2, Y \) such that

\[
\Pr(S = s, T = t, \dot{Q} = \dot{q}, X_1 = x_1, X_2 = x_2, Y = y) = p(s, t) p(q | \dot{q}) p(x_1 | \dot{q}, q) p(x_2 | \dot{q}, q) w(y | x_1, x_2).
\]

for which

\[
H(S | K) < I(X_1; Y | K \dot{Q}) \tag{4.11}
\]

\[
H(S | \dot{Q}) < I(SX_1; \dot{Q}), \tag{4.12}
\]

then the source \( (S, T) \) is \( S \)-admissible. Conversely, if the source \( (S, T) \) is \( S \)-admissible, the conditions (4.12), (4.13) with "<" replaced by "\( \leq \)" have to be satisfied for some \( \dot{Q}, X_1, X_2, Y \) satisfying (4.11). Here it is sufficient to consider only \( \dot{Q} \) such that \( ||\dot{Q}|| \leq ||X_1|| + 2 \).

**Proof:** See Appendix III.

**Remark 9:** If \( S = T = K \), conditions (4.12), (4.13) reduce to the optimal condition (3.32) in Example 3; for any distribution \( p(x_1, x_2) \) we can set \( \dot{Q} = X_1 X_2 \).
V. A Channel with Side Information at the Decoder

Shannon introduced in [21] a one sender-one receiver channel with finitely many states chosen from use to use independently at random and also independently of the letters sent (cf. channels with random state described in Section III). This channel can be viewed as being composed of a multiple-access channel with transmission probabilities \( w(y|x, t) \) for \( y \in \mathcal{Y}, x \in \mathcal{X}, t \in \mathcal{T} \) and a random mechanism selecting letters (states) \( t_1, t_2, \ldots \in \mathcal{T} \) that is a sequence of i.i.d. random variables \( T_1, T_2, \ldots \) with values in \( \mathcal{T} \). When neither the \( \mathcal{X} \)-sender nor the \( \mathcal{Y} \)-receiver know anything about the outcome of \( T_1, T_2, \ldots, T_n \) we are just dealing with an ordinary discretememoryless channel.

Shannon investigated a case of side information at the \( \mathcal{X} \)-sender: we know the outcomes of \( T_1, \ldots, T_n \) before we send \( X_{n+1} \). Wolfowitz [17] studied this and other cases.

Here it is assumed that the \( \mathcal{X} \)-sender has no side information, but that the \( \mathcal{Y} \)-decoder can be informed about the outcomes of \( T_1, \ldots, T_n \) via a separate noiseless channel at a fixed rate \( R \) (partial side information). Denote the capacities of this channel in dependence of \( R \) by \( C(R) \).

Conjecture: For \( R > 0 \),
\[
C(R) = \max \{ I(X; Y|V) | I(T; Y|V) \leq R, \quad V \rightarrow T \rightarrow Y, |V| < |T| + 1 \}.
\]

It is not hard to show by the argument exploited in Section III that the expression at the right-hand side is an achievable rate; the problem is to prove optimality, that is, a converse.

This channel is somewhat related to, but much simpler than, the interference channel (see [22], [23]) and represents one of the simplest cases in channel coding where the converse is still not known. A fairly simple example of this kind in source coding was formulated in [7, part II, sect. VII].

VI. A New Achievable Rate Region for the Binary Modulo-Two Sum Two Helper Problem

Let us consider a correlated source with two generic random variables \( X, Y, Z \) taking values in \( \mathcal{X} = \mathcal{Y} = \mathcal{Z} = \{0, 1\} \), respectively, such that \( Z = X \oplus Y \), where \( X \oplus Y = 0 \) if \( X = Y \), and 1 otherwise.

Suppose there are two encoders
\[
\phi_1: \mathcal{X}^n \rightarrow \mathcal{R}_1 = \{1, 2, \ldots, M_1\},
\]
\[
\phi_2: \mathcal{Y}^n \rightarrow \mathcal{R}_2 = \{1, 2, \ldots, M_2\}
\]
and the decoder
\[
\psi: \mathcal{R}_1 \times \mathcal{R}_2 \rightarrow \mathcal{Z}^n,
\]
which is required to reliably reproduce \( Z^n \) based on the knowledge of \( \phi_1(X^n) \) and \( \phi_2(Y^n) \).

Definition 4: \( (R_1, R_2) \) is said to be an achievable pair of rates, if for any \( 0 < \lambda < 1, 0 < \eta \) and all sufficiently large \( n \) there exist some \( \phi_1, \phi_2, \psi \) such that
\[
\frac{1}{n} \log ||\phi_1|| \leq R_1 + \eta,
\]
\[
\frac{1}{n} \log ||\phi_2|| \leq R_2 + \eta,
\]
and \( Pr(\hat{Z}^n + Z^n) \leq \lambda \), where \( Z^n = \psi(\phi_1(X^n), \phi_2(Y^n)) \), and \( ||\phi_|| \) denotes the cardinality of the range of the function \( \phi \).

Denote the closure of the set of all achievable pairs of rates by \( \mathcal{R}_\phi \). The problem of determining \( \mathcal{R}_\phi \) is a very special but also very interesting case of the two-helper side information problem stated by Ahlswede-Körner in [24]. Clearly, if the Slepian-Wolf conditions
\[
R_1 \geq H(H|Y),
\]
\[
R_2 \geq H(Y|X)
\]
\[
R_1 + R_2 \geq H(X, Y)
\]
(6.1)
are satisfied, then \( (R_1, R_2) \in \mathcal{R}_\phi \). Körner-Marton proved in [13] that \( (R_1, R_2) \in \mathcal{R}_\phi \), if
\[
R_1 \geq H(Z), \quad R_2 \geq H(Z).
\]
(6.2)
Moreover, they showed that (6.2) gives the exact rate region if \( Pr(\hat{Z} = 0) = Pr(\hat{Z} = 1) \) and \( Y \) is the output variable of a binary symmetric channel with input variable \( X \).

Here we present an achievable region \( \mathcal{R} \) which contains the regions described by (6.1) and (6.2), and is in general larger than \( \mathcal{R}_\phi \), the convex hull of both of them.

Let \( U, V \) be finite-valued random variables such that \( U, X, Y, V \) form a Markov chain in this order:
\[
U \rightarrow X \rightarrow Y \rightarrow V,
\]
(6.3)
and let \( \mathcal{R}(U, V) \) be the set of all \( (R_1, R_2) \) satisfying
\[
R_1 \geq I(U; X|V) + H(Z|UV),
\]
(6.4)
\[
R_2 \geq I(V; Y|U) + H(Z|UV),
\]
(6.5)
\[
R_1 + R_2 \geq I(UV; XY) + 2H(Z|UV).
\]
(6.6)
Denote by \( \mathcal{R} \) the convex closure of \( \bigcup_{U, V, \mathcal{R}(U, V)} \), where the union is taken over all \( U, V \) satisfying (6.3).

Theorem 10: \( \mathcal{R} \subset \mathcal{R}_\phi \).

The proof is given in Appendix IV, which is based on a combination of a standard technique in source coding [15] and the method of Elias [26] (cf. Gallager [20]) for finding linear codes, which was previously used by Körner-Marton for proving their result stated above.

Remark 10: Theorem 10 contains the previous results for the modulo-two sum problem. In fact, if we put \( U = X, \quad V = Y \) in (6.4)-(6.6) we obtain the Slepian-Wolf condition (6.1).

Next, if we put \( U = V = \phi \), we obtain the Körner-Marton condition (6.2).

The region \( \mathcal{R} \) established in Theorem 10 strictly extends \( \mathcal{R}_\phi \) for general binary sources, as is seen from the following example.
Example 4: We choose $X, Y$ with the probability distribution:

\[
\begin{align*}
\Pr(X = 0, Y = 0) &= 0.003920, \\
\Pr(X = 0, Y = 1) &= 0.976080, \\
\Pr(X = 1, Y = 0) &= 0.015920, \\
\Pr(X = 1, Y = 1) &= 0.000080; \\
\Pr(X = 0) &= 0.980090, \quad \Pr(Y = 0) = 0.023840, \\
\Pr(Z = 0) &= 0.004000,
\end{align*}
\]

for which we have in bits

\[
\begin{align*}
H(X) &= 0.1414405, \quad H(Y) = 0.1624894, \\
H(Z) &= 0.0376223, \quad H(XY) = 0.1790629.
\end{align*}
\]

Choose auxiliary variables $U, V$ taking values in $(0, 1)$ with

\[
\begin{align*}
\Pr(U = 0 | X = 0) &= 0.55, \quad \Pr(U = 0 | X = 1) = 0.45, \\
\Pr(V = 0 | Y = 0) &= 0.95, \quad \Pr(V = 0 | Y = 1) = 0.05,
\end{align*}
\]

then the point $P = (R_1, R_2)$ with

\[
\begin{align*}
R_1 &= I(U; X) + H(Z | UV), \\
R_2 &= I(V; Y | U) + H(Z | UV)
\end{align*}
\]

has the value $R_1 = 0.0251236$, $R_2 = 0.1096321$. The region $\mathcal{R}$ is illustrated in Fig. 6, in which the boundary line $AB$ is specified by

\[
\begin{align*}
&f(R_1, R_2) = (R_1 - H(Z))(H(Y) - H(Z)) \\
&\quad + (R_2 - H(Z))(H(X) - H(X | Y)) = 0.
\end{align*}
\]

For the point $P$ we have

\[
f(R_1, R_2) = -0.000043
\]

which implies that at least the point $P$ lies outside $\mathcal{R}_0$.

VII. CORRELATED CODES: AN ALTERNATE APPROACH TO THE PROBLEM OF TRANSMITTING CORRELATED MESSAGES

We first take a closer look at the quote from [1] given in the Introduction and emphasize the following observations.

a) In sentence 1) Shannon asks whether it is always possible to achieve the outer bound $G_0$ to the capacity region of the two-way channel for messages with a suitable dependency structure.

b) Sentence 4) seems to indicate that he has in mind that the message structure is that of a memoryless correlated source, but in sentence 1) the message structure is not specified.

c) In sentence 2) Shannon assigns rates $R_1, R_2$ to whatever "source-channel-code" he has in mind. It seems to us that only with two parameters such as $R_1, R_2$ no reasonable code of this kind can be satisfactorily described. Note that Cover-El Gamal-Salehi avoid talking about rates at all in their approach (Definition 1, also cf. Han [27]).

d) It is by now well-known that in the problem of transmitting correlated messages over multiway channels the classical separation principle of Shannon (separate coding of the source and the channel for single-user channels) fails in the sense that separate coding results in loss of efficiency (in rates, if defined). The approach indicated in [1] as well as the particular "source-channel-code" (or matching) chosen in [6] take this into account. An obvious drawback of such an approach is that it is only meaningful if the message statistics are known exactly. We tend to share the opinion expressed by Gallager in [20, p. 14]: "In many data transmission systems the probabilities with which messages are to be used are either unknown or meaningless." Therefore the results of [6] and also our results in the previous sections should probably be considered to be more of the nature of providing theoretical insight than of direct practical importance. The least one should try is to safeguard against classes of message distributions (see [25] and [7]). Another way of coming closer to a real communication situation with our models consists of enforcing the separation principle (in spite of its suboptimality in an ideal situation) and investigating what can be done (also optimally) if source and channel coding are carried out separately.

A first approach is to study correlated codes for channels without any reference to sources. In particular it is interesting to know whether the question raised by Shannon, namely, the achievability of the outer bound, can be formulated for correlated codes and whether then the answer is positive. We shall focus here on the multiple-access channel, because it is better understood than the two-way channel. We find it now more convenient to denote the input alphabets by $\mathcal{X}, \mathcal{Y}$, and the output alphabet by $\mathcal{Z}$.

Definition 5: An $(n, M_x, M_y, M_z, \alpha, \lambda)$ correlated code for the MAC $M_2$ is a system $(\mathcal{R}_x, \mathcal{R}_y, \alpha, \lambda)$, where

\[
\begin{align*}
\mathcal{R}_x &= \{ u_1, \ldots, u_{M_x} \} \subset \mathcal{X}^n, \quad |\mathcal{R}_x| = M_x \\
\mathcal{R}_y &= \{ v_1, \ldots, v_{M_y} \} \subset \mathcal{Y}^n, \quad |\mathcal{R}_y| = M_y \\
\mathcal{R}_z &= \{ s_1, \ldots, s_{M_z} \} \subset \mathcal{Z}^n, \quad |\mathcal{R}_z| = M_z \\
\alpha: \mathcal{R}_x \times \mathcal{R}_y \to (0, 1); \quad \sum_{i=1}^{M_x} \sum_{j=1}^{M_y} \alpha(u_i, v_j) = M_{xy}
\end{align*}
\]
\( \mathcal{D} = \{ D_{ij} \in \mathbb{Z}^n | \alpha(u_i, v_j) = 1 \} \),
\( D_{ij} \cap D_{ij'} = \phi \), for \( (i, j) \neq (i', j') \),
\( w(D_{ij} | u_i, v_j) \geq 1 - \lambda \), for every \( u_i, v_j \)
with \( \alpha(u_i, v_j) = 1 \). (7.5)

Other quantities of interest are
\( M_{iij} = \sum_i \alpha(u_i, v_j) M_{iij} = \sum_i \alpha(u_i, v_j) \),
\( M_{iij} = \frac{1}{M} \sum_i M_{iij} = M_{iij} / M \),
\( M_{iij} = \frac{1}{M} \sum_i M_{iij} = M_{iij} / M \),
\( M_{iij} = \frac{1}{M} \sum_i M_{iij} = M_{iij} / M \),
\( M_i = M_{iij} \left( M_{iij} M_{iij} \right)^{-1} = M_{iij} M_{iij} = M_{iij} M_{iij} \). (7.7)
\( M_{0} = M_{iij} \left( M_{iij} M_{iij} \right)^{-1} = M_{iij} M_{iij} = M_{iij} M_{iij} \). (7.8)

Clearly, \( M_{iij}, M_{iij}, \) and \( M_{iij}, \) uniquely determine \( M_{iij}, M_{iij}, \) and \( M_{iij}, \) and vice versa.

**Definition 6:** An \( \left( n, M, M_{iij}, M_{iij}, \alpha, \lambda \right) \) correlated side-information code for \( M_{iij} \) is a system satisfying (7.1)–(7.3) and
\( \mathcal{D} = \{ D_{ij} \subset \mathbb{Z}^n | i \leq M \} \),
\( D_{ij} \cap D_{ij'} = \phi \), for \( i \neq i' \),
\( w(D_{ij} | u_i, v_j) \geq 1 - \lambda \), for every \( u_i, v_j \)
with \( \alpha(u_i, v_j) = 1 \). (7.9)

This code concept is for the case where only the messages in \( \mathbb{R} \) are to be reproduced. Note that these code definitions do not necessarily imply that \( u_i = u_i \), for \( i = i' \) (resp. \( v_j = v_j \), for \( j = j' \)).

To the parameters \( M_{iij}, M_{iij}, M_{iij}, M_{iij}, M_{iij} \), we can assign the rates \( R_{iij} = (1/n) \log M_{iij}, R_{iij} = (1/n) \log M_{iij}, R_{iij} = (1/n) \log M_{iij}, R_{iij} = (1/n) \log M_{iij} \). Clearly, \( R_{iij} + R_{iij} \leq R_{iij} \), \( R_{iij} \). \( R_{iij} \).

**Definition 7:** \( \left( R_{iij}, R_{iij}, R_{iij} \right) \) is an achievable rate-triple for the correlated code problem (resp. correlated side-information code problem) for \( M_{iij} \), if for any \( \eta > 0, 0 < \lambda < 1 \), and every sufficiently large \( n \) there exists for some \( R_{iij}, R_{iij}, R_{iij} \) such that \( \left| R_{iij} - R_{iij} \right| \leq \eta, \left| R_{iij} - R_{iij} \right| \leq \eta, \left| R_{iij} - R_{iij} \right| \leq \eta \), an \( \left( n, \alpha(n) \log M_{iij}, \alpha(n) \log M_{iij}, \alpha(n) \log M_{iij}, \alpha(n) \log M_{iij} \right) \) correlated code (resp. correlated side-information code). Denote the region of achievable rate triples \( \left( R_{iij}, R_{iij}, R_{iij} \right) \) by \( \mathcal{R}_{iij} \) (resp. \( \mathcal{R}_{iij} \)).

**A Partial Result for \( \mathcal{R}_{iij} \)**

Using the identities
\( R_{iij} = R_{iij} - R_{iij}, R_{iij} = R_{iij} - R_{iij} \), (7.11)
one can equivalently transform \( \mathcal{R}_{iij} \) into the set \( \mathcal{R}_{iij} \) of achievable triples \( \left( R_{iij}, R_{iij}, R_{iij} \right) \), and vice versa. It is often more convenient to work with \( \mathcal{R}_{iij} \).

Define \( \mathcal{R}^* \) as the set of triples \( \left( R_{iij}, R_{iij}, R_{iij} \right) \) satisfying
\( R_{iij} = I(X; Z|XY), R_{iij} = I(Y; Z|X), \)
\( R_{iij} = I(X; Z|Q), \)
for some correlated input variable pair \( (X, Y) \) and the corresponding output variable \( Z \), where \( Q \) is the time-sharing parameter. With a slight modification of the standard argument used to establish the coding theorem on MAC, we can easily prove the following theorem.

**Theorem 11:**
\( \mathcal{R}_{iij} \supset \mathcal{R}^* \). (7.12)

It should be noted that the result of Slepian-Wolf [5] for the MAC with correlated sources can be viewed as a coding theorem for correlated codes with an \( a \) having the 1 in disjoint equal-sized rectangles placed along a diagonal. For those \( a \) they show that \( \left( R_{iij}, R_{iij}, R_{iij} \right) \) is achievable if
\( R_{iij} \leq I(X; Z|Y), \)
\( R_{iij} \leq I(Y; Z|X), \)
\( R_{iij} \leq I(X; Z), \)
for some \( Q \), \( X, Y, Z \) with \( X \to Q \to Y, Q \to XY \to Z \). This region is in general smaller than \( \mathcal{R}_{iij} \) for the multiple-access channel.

**B. Remarks about \( \mathcal{R}_{iij} \)**

In this case of side codes the achievability of \( \left( R_{iij}, R_{iij}, R_{iij} \right) \) implies that \( \left( R_{iij}, R_{iij}, R_{iij} \right) \) is also achievable for \( c > 1 \), because one can always take the \( q \), with multiplicity. Therefore one should look for the smallest \( R_{iij} \) such that the triple is achievable.

One may also study first the projection \( \left( R_{iij}, R_{iij} \right) \). This already constitutes an interesting new problem. Let \( \mathcal{R}_{iij} \) be the region of those achievable pairs.

Obviously, \( \left( 0, 0 \right), \max_{c} I(X; Z|Y), \max_{c} I(X; Z|Y), \max_{c} I(X; Z|Y), \) \( \max_{c} I(X; Z|Y), \) and their convex combinations are in \( \mathcal{R}_{iij} \).

**Problem 1:** What is the exact region of \( \mathcal{R}_{iij} \) (resp. of its projection)?

With this short discussion we hope to have made clear that there are several new problems for the code concepts defined (and others one might think of) if used for the MAC, and there is a whole collection of problems if one considers various multiway channels such as the TWG, broadcast channel, interference channel, etc. Instead of going into further details, we now give some ideas about source-channel matching.
C. The Subcode and Minor Problem

The following problems strikingly demonstrate the richness of multi-user information theory in comparison to the classical single-user theory. First notice that for the two code concepts defined above one can already pass from average error to maximal codes without essential loss in rate and hence the rate (capacity) regions are the same. We can therefore pass from a code \( (\mathcal{M}_x, \mathcal{M}_y, \alpha, \overline{b}) \) to a (minor) subcode \( (\mathcal{M}^a_x, \mathcal{M}^a_y, \alpha^a, \overline{b}^a) \) without increase in error probability, where \( \mathcal{M}^a_x \subseteq \mathcal{M}_x, \mathcal{M}^a_y \subseteq \mathcal{M}_y, \alpha^a \) is the restriction of \( \alpha \) on \( \mathcal{M}^a_x \times \mathcal{M}^a_y \), and \( \overline{b}^a \) is a new decoding rule, which can, but need not, be obtained by restricting \( \overline{b} \).

Denote the collection of all such subcodes by \( \mathcal{C}(\mathcal{M}_x, \mathcal{M}_y, \alpha, \overline{b}) \).

A similar but larger class of subcodes is the collection of all subcodes, denoted by \( \mathcal{C}(\mathcal{M}_x, \mathcal{M}_y, \alpha, \overline{b}) \), obtained if we replace \( \alpha^a \) in the above by any \( \alpha^a \) with \( \alpha^a \preceq \alpha \), where \( \preceq \) means that \( a(u, v) = 1 \) whenever \( a^a(u, v) = 1 \). Clearly, \( \mathcal{C}(\mathcal{M}_x, \mathcal{M}_y, \alpha, \overline{b}) \subseteq \mathcal{C}(\mathcal{M}_x, \mathcal{M}_y, \alpha, \overline{b}) \).

Assigning rates \( R_x^a, R_y^a, R_z^a \) to these subcodes, one obtains two spectra of rate triples from a single correlated code, which we denote by \( \mathcal{S}_C(\mathcal{M}_x, \mathcal{M}_y, \alpha, \overline{b}) \) and \( \mathcal{S}_C(\mathcal{M}_x, \mathcal{M}_y, \alpha, \overline{b}) \), respectively.

**Problem 2:** Which spectra \( \mathcal{S}_C^* \) (resp. \( \mathcal{S}_C \)) are achievable with arbitrarily small error probabilities for \( n \) large?

Note that the same correlated code can be used for several message sources by suitably selecting its subcodes (in this sense it may be called a multi-user code) and the richness of the spectrum \( \mathcal{S}_C^* \) (resp. \( \mathcal{S}_C \)) is a criterion for its capability to meet various demands.

Some hard problems arise. For instance, to what extent does the spectrum of a code allow to reconstruct this code (at least in an approximate sense)? There is a relation to reconstruction problems of bipartite graphs from subgraphs, because both problems are based on the same kind of incidence structure \( \alpha \).

D. Matching

An abstract correlated source is simply a pair of random variables \( (S, T) \) where \( S, T \) take values in finite sets \( S \), resp. \( T \).

We say that this source can be \( \alpha \)-matched with the correlated code \( (\mathcal{M}_x, \mathcal{M}_y, \alpha, \overline{b}) \) if there exist functions \( f: S \to \mathcal{M}_x, g: T \to \mathcal{M}_y, \psi: \mathcal{M}_x \times \mathcal{M}_y \to S \times T \) such that

\[
\Pr(f(S), g(T)) = (S, T) \geq 1 - \epsilon \quad (7.17)
\]

and

\[
\Pr(\alpha(f(S), g(T)) = 1) \geq 1 - \epsilon. \quad (7.18)
\]

Now imagine that the component sources \( S, T \) are not directly placed at the channel \( C \)-encoder resp. \( D \)-encoder, but are linked to them, respectively, by the channel which may be assumed to be noiseless. Then the cardinalities of the ranges of \( f \) and \( g \) become an issue, because the channel can transmit only at limited rates \( R_1 \) resp. \( R_2 \). We say that \((S, T)\) can be \((R_1, R_2, \epsilon)\)-matched with \((\mathcal{M}_x, \mathcal{M}_y, \alpha, \overline{b})\) if, in addition to (7.17), (7.18), also \( \|f\| \leq R_1, \|g\| \leq R_2 \). Finally, we say that \((S, T)\) is \((R_1, R_2, \epsilon, \lambda)\)-transmissible over the channel \( M_1 \), if it can be \((R_1, R_2, \epsilon)\)-matched with some correlated code \((\mathcal{M}_x, \mathcal{M}_y, \alpha, \overline{b})\).

**Remark 11:** Cover, El Gamal, and Salehi [6] assumed in their approach that the source outputs of the component sources are available at the corresponding channel encoders. An interesting more general problem arises if, as above, the outputs of the component sources are to be transmitted via channels of rates \( R_1 \), resp. \( R_2 \), to the corresponding encoders of \( M_2 \).

E. Continuous Transmission and the Embedding Problem

Suppose that the correlated code \((\mathcal{M}_x, \mathcal{M}_y, \alpha, \overline{b})\) is to be used repeatedly. If we place a probability distribution \( \beta \) on \( \mathcal{M}_x \times \mathcal{M}_y \), such that for instance

\[
\beta(u, v) = \left( \sum_{i' \neq i} \alpha(u, v) \right)^{-1} \quad \text{if} \quad \alpha(u, v) = 1,
\]

then we obtain a correlated code \((\mathcal{M}_x, \mathcal{M}_y, \alpha, \overline{b})\) with an average (taken with respect to \( \beta \)) error probability less than \( \lambda \).

If we denote by \( I, J \) random variables with values in \( \mathcal{M}_x \) resp. \( \mathcal{M}_y \) and joint distribution \( \beta \), then the repeated use of the code can be described by \((I, J)_{n=1}^n\) where the \((I, J)\) are independent copies of \((I, J)\). For simplicity let us assume that the message statistic is that of a correlated source \((S, T)_{n=1}^n\), which is to be transmitted and reproduced according to some fidelity criterion. This situation is quite complex and as a first step for its analysis one may consider the following problem, which is also of interest in itself.

**Embedding Problem:** For two (correlated) sources \((X_t, Y_t)_{t=1}^n\), \((U_t, V_t)_{t=1}^n\) and \(0 < \epsilon < 1\) it is to be decided whether there exist functions \( f: \Delta_x \to \mathcal{M}_x, g: \Delta_y \to \mathcal{M}_y, \psi: \mathcal{M}_x \times \mathcal{M}_y \to \Delta_x \times \Delta_y \) with

\[
\Pr(\psi(f(X^e), g(Y^e)) = (X^e, Y^e)) \geq 1 - \epsilon, \quad (7.19)
\]

and

\[
\sum_{(u^m, v^m) \in (f(X^e) \times g(Y^e))} |Pr_f(u^m, v^m) - Pr_g(u^m, v^m)| < \epsilon,
\]

where

\[
Pr_f(u^m, v^m) = \Pr((f(X^e), g(Y^e)) = (u^m, v^m)),
\]

\[
Pr_g(u^m, v^m) = \frac{\Pr((U^m, V^m) = (u^m, v^m))}{\Pr((U^m, V^m) \in (X^e) \times (Y^e))}.
\]

Further, what are the minimal rates of \( f \) and \( g \)?

**APPENDIX I**

**Proof of Theorem 4**

As will be seen from the way of generating the random codes in the proof of Theorem 3, the admissibility of \( S \) based on
\[ r(\lambda) \geq H(S^n | Y^n X_2^n) = \sum_{i=1}^{n} H(S_i | Y^n X_2^n \tilde{S}^{i-1}) \geq \sum_{i=1}^{n} H(S_i | Y^n X_2^n T^{-1} \tilde{S}^{i-1}) \]

\[ = \sum_{i=1}^{n} H(S_i | Y_i X_2^n T^{-1} \tilde{S}^{i-1}) = \sum_{i=1}^{n} H(S_i | Y_i X_2^n) , \]

where equality (1) follows from the fact that \( Y_i^{-1} \tilde{S}^{i+1} \), \( X_2^n T^{-1} \tilde{S}^{i+1} \), \( S_i \) from a Markov chain in this order, indicated as \( Y_i^{-1} \tilde{S}^{i+1} \rightarrow X_2^n T^{-1} \tilde{S}^{i+1} \rightarrow S_i \), which holds because \( \phi_i \) is componentwise.

Since \( V_i \) contains \( X_2^n \), we have

\[ \sum_{i=1}^{n} H(S_i | V_i) \leq \sum_{i=1}^{n} H(S_i | V_i , X_2^n) \]

\[ - \sum_{i=1}^{n} H(S_i | Y_i X_2^n) + r(\lambda) \]

\[ = \sum_{i=1}^{n} I(S_i ; Y_i X_2^n) + r(\lambda) \]

\[ \leq \sum_{i=1}^{n} I(S_i ; X_i , Y_i X_2^n) + r(\lambda) \]

\[ = \sum_{i=1}^{n} I(X_i ; Y_i X_2^n) + r(\lambda) , \]

where in the last step we have used the memoryless character of the channel. Next, noting that \( Y_0 \) contains \( X_2^n \) by assumption, we have \( H(S^n X_2^n | Y_0^n) \leq r(\lambda) \). Hence,

\[ H(S^n X_2^n) \leq H(S^n X_2^n) - H(S^n X_2^n | Y_0^n) \]

\[ = I(S^n X_2^n ; Y_0^n) - H(Y_0^n) + H(S^n X_2^n) \]

\[ \leq \sum_{i=1}^{n} I(S_i X_i ; Y_i) - 2 H(Y_i) \]

\[ \leq \sum_{i=1}^{n} H(Y_i) - \sum_{i=1}^{n} H(Y_i | S^n X_2^n Y_0^{-1}) \]

\[ = \sum_{i=1}^{n} H(Y_i) - \sum_{i=1}^{n} H(Y_i | S^n X_2^n Y_0^{-1}) \]

\[ = \sum_{i=1}^{n} I(X_i ; Y_i | X_2^n) . \]

Proof of Theorem 5

1) The Direct Part: Replace \( V \) by \( V X_2 \) in (3.2), (3.3) and use the relations

\[ I(S X_2 ; Y X_2|V Q) = I(X_2; Y | X_2 Q) . \]

Note that the replacement of \( V \) by \( V X_2 \) does not violate the form of condition (3.26) because \( p(x_2 | v, q) \) is not of the form which depends on \( t \).

2) The Converse Part: We use the same notation as that in the proof of Theorem 4 (Appendix I). Set \( V_i = X_2^n T^{-1} \tilde{S}^{i-1} \), where \( \tilde{S}^{i-1} = (S_1, \cdots , S_{i-1}) \). Then, by Fano's lemma.

\[ \sum_{i=1}^{n} H(S_i | V_i) + \sum_{i=1}^{n} I(S_i ; T^{-1} X_2^n \tilde{S}^{i-1}) \leq \sum_{i=1}^{n} H(S_i | V_i , X_2^n) + \sum_{i=1}^{n} I(S_i ; T^{-1} X_2^n \tilde{S}^{i-1}) . \]

APPENDIX II
where $T^{[i]} = (T_1, \cdots, T_{i-1}, T_{i+1}, \cdots, T_n)$. Hence, defining $Q, Q', S, T, X_1, X_2, Y$ by

$$\Pr(Q = i) = 1/n, \quad i = 1, \cdots, n;$$

then $S = S_i, \ T = T_i, \ K = K_i, \ Q' = T^{[i]}, \ X_1 = X_i, \ X_2 = X_i, \ Y = Y_i$, given $Q = i$,

we have

$$H(S|K) - \frac{1}{n} r(\lambda) \leq I(X_i; Y|KQ'Q). \quad (A6)$$

On the other hand,

$$H(S^*) - r(\lambda) \leq I(S^*; Y^*) = I(S^*X_i; Y^*)$$

$$= H(Y^*) - H(Y|S^*X_i)$$

$$\leq \sum_{i=1}^{n} H(Y_i) - \sum_{i=1}^{n} H(Y_i|S^*X_i Y^{[i]})$$

$$= \sum_{i=1}^{n} H(Y_i) - \sum_{i=1}^{n} H(Y_i|S^*X_i T^{[i]})$$

$$= \sum_{i=1}^{n} H(Y_i) - \sum_{i=1}^{n} H(Y_i|S_i X_i T^{[i]})$$

$$= \sum_{i=1}^{n} I(S_i X_i T^{[i]}; Y_i)$$

$$= nI(SX; Q'; Y|Q') \leq nI(SX; Q'; Y). \quad (A7)$$

Hence,

$$H(S) - \frac{1}{n} r(\lambda) \leq I(SX; Q'; Y). \quad (A7)$$

Inequalities (A6), (A7) with $\hat{Q} = Q'$ yield the latter part. Q.E.D.

**APPENDIX IV**

**Proof of Theorem 10**

By the time-sharing argument it suffices to show $R(U, V) \in R_{\phi}$ for any $(U, V)$ and for this we have to consider only the extreme points of $R(U, V)$, say,

$$R_1 = I(U; X) + H(Z|UV), \quad (A8)$$

$$R_2 = I(V; Y|U) + H(Z|UV). \quad (A9)$$

Decompose the rates $(R_1, R_2)$ as follows:

$$R_1 = R_1^{[1]} + R_1^{[2]}, \quad R_2 = R_2^{[1]} + R_2^{[2]}, \quad (A10)$$

where

$$R_1^{[1]} = I(U; X), \quad R_1^{[2]} = H(Z|UV), \quad (A11)$$

$$R_2^{[1]} = I(V; Y|U), \quad R_2^{[2]} = H(Z|UV). \quad (A12)$$

We use the rates $(R_1^{[1]}, R_1^{[2]})$ in the first step, and the rates $(R_2^{[1]}, R_2^{[2]})$ in the second step.

**Step 1:** Denote by $\tilde{U}_1, \cdots, \tilde{U}_n$ mutually independent random variables distributed uniformly in $T_i(U)$, and by $\tilde{V}_1, \cdots, \tilde{V}_n$ mutually independent random variables distributed uniformly in $T_i(V)$, where $\eta > 0$ is a positive number, and

$$L_1 = \exp \left[ n \left( I(U; X) + \eta/2 \right) \right],$$

$$= \exp \left[ n \left( R_1^{[1]} + \eta/2 \right) \right], \quad (A13)$$

$$J = \exp \left[ n \left( I(V; Y) + \eta/4 \right) \right]. \quad (A14)$$
Then, there exist functions \( U^* = U^*(X^*, \bar{U}_1, \ldots, \bar{U}_J) \) \( V^* = V^*(Y^*, \bar{V}_1, \ldots, \bar{V}_J) \) such that \( U^* = \bar{U}_i \) for some \( i = 1, \ldots, L_1 \); and \( V^* = \bar{V}_i \) for some \( i = 1, \ldots, L_2 \); and for sufficiently large \( n \)

\[
\text{Pr} \left( (U^*, V^*, X^*, Y^*) \in T_i(UVXY) \right) > 1 - \delta_1 , \quad (A15)
\]

where \( \delta_1 = \delta_1(\epsilon, \nu) \rightarrow 0 \) as \( \epsilon \rightarrow 0 \) (cf. [16]).

Put \( \bar{E}_i = (1, 2, \ldots, L_1) \), \( \bar{E}_j = (1, 2, \ldots, L_2) \) where \( L_2 = \exp [n(R_1'^* + \eta_2/2)] \), and partition \( (\bar{V}_1, \ldots, \bar{V}_J) \) into \( L_2 \) subclasses of the same cardinality. Let us define the subencoders \( h_1^{(i)}: X^* \rightarrow \bar{E}_i, h_2^{(i)}: Y^* \rightarrow \bar{E}_i \) by \( h_1^{(i)}(X^*) = i \) if \( U^* = \bar{U}_i \); and by \( h_2^{(i)}(Y^*) = i \) if \( V^* \) belongs to the \( i \)th subclass of \( (\bar{V}_1, \ldots, \bar{V}_J) \). The corresponding subencoder

\[
k: \bar{E}_i \times \bar{E}_j \rightarrow (\bar{U}_1, \ldots, \bar{U}_J) \times (\bar{V}_1, \ldots, \bar{V}_J) \quad (A16)
\]

is defined by \( k(i, j) = (\bar{U}_i, \bar{V}_j) \) if \( \bar{U}_i \) is one and only one element of the \( j \)th subclass such that \( (\bar{U}_i, \bar{V}_j) \in T_i(UVY) \); otherwise let \( k(i, j) \) be an arbitrary element. Then, by virtue of the standard evaluation technique in multiterminal source coding.

\[
\text{Pr} \left( k(i, j) = (U^*, V^*) \right) \left( U^*, V^*, X^*, Y^* \right) \in T_i(UVXY) \right) \geq 1 - \delta_1 , \quad (A17)
\]

where \( \delta_1 = \delta_1(\epsilon, \nu) \rightarrow 0 \) as \( \epsilon \rightarrow 0 \).

**Step 2:** Let \n
\[
m = n \left( \frac{H(Z|UV) + \eta/2}{\log 2} \right) \quad (A18)
\]

and consider the random linear mapping \( f: \{0, 1\}^n \rightarrow \{0, 1\}^n \) such that \( f(z) = Az \), where \( A \) is an \( m \times n \) matrix whose elements are all independently and uniformly distributed in \( (0, 1) \). Here we regard \( (0, 1) \) as forming the Galois field \( \mathbb{GF}(2) \).

Denote other subencoders \( h_1^{(1)}: X^* \rightarrow \bar{E}_1, h_2^{(1)}: Y^* \rightarrow \bar{E}_1 \) by \( h_1^{(1)}(X^*) = f(X^*), h_2^{(1)}(Y^*) = f(Y^*) \). Note here that

\[
(1/n) \log |\mathbb{R}| - R_1'^* + \eta/2 = R_1'^* - \eta/2 . \quad (A19)
\]

**Step 3:** Let us now define the composite encoders \( \phi_1: X^* \rightarrow \bar{E}_1 \times \bar{E}_1 \), \( \phi_2: Y^* \rightarrow \bar{E}_2 \times \bar{E}_2 \) by \( \phi_1 = (h_1^{(1)}, h_2^{(1)}), \phi_2 = (h_1^{(1)}, h_2^{(1)}) \), and the composite decoder \( \psi: \bar{E}_1 \times \bar{E}_2 \rightarrow \mathbb{Z} \) by \( \psi(i, j, f(X^*), f(Y^*)) = z \) if \( z \) is one and only element of \( T_i(Z|k(i, j)) \) such that \( f(z) = f(X^*) + f(Y^*) \neq f(Z^*) \), where \( i \in \bar{E}_1, j \in \bar{E}_2 \).

To evaluate the probability of decoding error, consider the following error event:

\[ E_1: (U^*, V^*, X^*, Y^*) \in T_i(UVXY), \]

\[ E_2: (k(i, j)) = (U^*, V^*), \]

\[ E_3: (k(i, j)), Z^* \in T_i(UVZ), \]

\[ E_4: f(z) = f(Z^*), \quad \text{for some } z = Z^* \text{ such that } \quad (k(i, j)), z \in T_i(UVZ), \]

As the events \( E_1 \cap E_2 \cap E_3 \cap E_4 \) imply the event \( E_3 \), so that

\[
\text{Pr} \left( E_1 \cap E_2 \cap E_3 \cap E_4 \right) = 0 . \quad (A22)
\]

On the other hand,

\[
\text{Pr} \left( \bigcap_{i, j} E_i \cap E_2 \cap E_3 \right) \leq \sum_{i,j} \left[ \text{Pr} \left( f(z) = f(Z^*) \cap E_i \cap E_2 \cap E_3 \right) \right] , \quad (A23)
\]

where the random variables in (A23) are \( A \) in addition to \( X^*, Y^*, \bar{U}_1, \bar{V}_1 \).

As all the elements of \( A \) are independently and uniformly distributed in \( (0, 1) \), by counting all the cases satisfying \( f(z) = f(Z^*) \) it follows that for any \( z \neq Z^* \)

\[
\text{Pr} \left( f(z) = f(Z^*) \cap E_i \cap E_2 \cap E_3 \right) = (2^{-n})/2^m = 2^{-m} . \quad (A23)
\]

Hence, by (A18),

\[
\text{Pr} \left( E_4 \cap E_1 \cap E_2 \cap E_3 \right) \leq T_i (Z|UV) \cdot 2^{-m} . \quad (A24)
\]

Finally, note that the total rates \( (R_1, R_2) \) used for the composite encoders \( \phi_1, \phi_2 \) are

\[
R_1 = R_1'^* + R_2'^* + \eta - R_1 + \eta , \quad R_2 = R_1'^* + R_2'^* + \eta = R_2 + \eta , \quad (A25)
\]

and hence the rate given by (A8) and (A9) is achievable. Q.E.D.

**References**


Estimation of Spatial and Spectral Parameters of Multiple Sources

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Abstract—The problem of estimating spatial and spectral parameters of multiple sources by sensor arrays is considered. A parametric model is derived for the multichannel measurement vector. The model parameters are the time delays and the autoregressive moving-average parameters of all the sources. A subspectral parameter estimation procedure is proposed, and simulation results are presented to illustrate its performance.

I. INTRODUCTION

PASSIVE surveillance systems are often required to detect sources of acoustic or electromagnetic energy and to estimate their location and spectral characteristics. Examples include sonar systems, acoustic arrays for detection of low-flying aircraft, and geophones for detection and localization of seismic events. These systems consist of multiple sensors arranged in some pattern such as a linear array. The relative time delays between the arrival of signals from a point source to the various sensors contain information about the location (bearing and range) of the source. For moving sources and sensors additional location information is contained in the differential Doppler shifts of the signals arriving at different sensors.

The estimation of time delay and Doppler shift has been studied extensively in the past two decades. The subject of optimum (or maximum-likelihood) processing has received particular attention [1]–[4]. The structure of the maximum-likelihood estimator (MLE) of delay and Doppler has been developed for various situations. Some of the key results are discussed briefly in the following paragraphs.

The MLE for the time delay between signals arriving at two sensors (with no differential Doppler shift) can be implemented by a generalized correlation procedure [5]: the sensor data are filtered and then cross-correlated; the location of the peak of the cross-correlation function provides the delay estimate. The optimal filters depend on the spectral characteristics of the signal and noise and on the signal-to-noise ratio (SNR).

When more than two sensors are available it is possible to combine the pairwise delay estimates (obtained by generalized cross correlation) to get a global delay estimate [6]. An alternative implementation of the delay estimator is provided by a conventional beamformer configuration involving a set of steering delays followed by filtering.