The Relationship between Hadamard and Conventional Multiplication for Positive Definite Matrices*

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ABSTRACT

It is known that if A is positive definite Hermitian, then $A\cdot A^{-1}\geqslant I$ in the positive semidefinite ordering. Our principal new result is a converse to this inequality: under certain weak regularity assumptions about a function F on the positive definite matrices, $A\cdot F(A)\geqslant AF(A)$ for all positive definite A if and only if $F(A)$ is a positive multiple of A^{-1}. In addition to the inequality $A\cdot A^{-1}\geqslant I$, it is known that $A\cdot A^{-1T}\geqslant I$ and, stronger, that $\lambda_{\min}(A\cdot B)\geqslant\lambda_{\min}(AB^T)$, for A, B positive definite Hermitian. We also show that $\lambda_{\min}(A\cdot B)\geqslant\lambda_{\min}(AB)$ and note that $\lambda_{\min}(AB)$ and $\lambda_{\min}(AB^T)$ can be quite different for A, B positive definite Hermitian. We utilize a simple technique for dealing with the Hadamard product, which relates it to the conventional product and which allows us to give especially simple proofs of the closure of the positive definites under Hadamard multiplication and of the inequalities mentioned.

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Let P_n denote the set of n-by-n positive definite Hermitian matrices, while $\bar{P}_n$ denotes its closure, the positive semidefinite matrices. As is conventional, we take P_n and $\bar{P}_n$ to be partially ordered via

$$B \geqslant A \quad \text{if and only if} \quad B - A \in \bar{P}_n$$

and

$$B > A \quad \text{if and only if} \quad B - A \in P_n,$$

the positive semidefinite and positive definite orderings, respectively.

The *Hadamard* (or entrywise) *product* [5] of two matrices $A = (a_{ij})$ and $B = (b_{ij})$ of the same dimensions is denoted and defined by

$$A \cdot B = \left(a_{ij} b_{ij}\right),$$

while conventional matrix multiplication is indicated, as usual, by juxtaposition.

It is often attributed to Schur and has long been known [5] that P_n (and $\bar{P}_n$) is closed under the Hadamard product.

THEOREM 1. *If $A, B \in P_n$ (respectively $\bar{P}_n$), then $A \cdot B \in P_n$ (respectively $\bar{P}_n$).*

(Furthermore, it is easily observed that if $A \in P_n$ and $B \in \bar{P}_n$, then $A \cdot B \in P_n$, unless B has a diagonal entry equal to 0.) Of course, P_n is *not* closed under conventional multiplication, but $A, B \in P_n$ does imply that the eigenvalues of AB are positive real numbers [4] and thus that $AB \in P_n$ if A and B commute.

It was first noted by Fiedler [1, 2] that

THEOREM 2a. *If $A \in P_n$, then*

$$A \cdot A^{-1T} \geqslant I.$$

This inequality cannot be strict; for, letting $e = (1, 1, \dots, 1)^T$, one has $(A \cdot A^{-1T})e = e$, so that 1 is an eigenvalue of $A \cdot A^{-1T}$.

By a very different proof, and without a clear realization that it was a different inequality (the comments pertain to the real case, but the proof works in the complex case), it was noted in [6] that

THEOREM 2b. *If* $A \in P_n$, *then*

$$A \cdot A^{-1} \geqslant I.$$

It is possible for this inequality to be strict when A is complex. (This is contrary to the statement made in [6] under the implicit assumption that A is real.) For example, let

$$A = \begin{bmatrix} 3 & 1-i & -i \\ 1+i & 2 & 1 \\ i & 1 & 1 \end{bmatrix}.$$

Of course Theorems 2a and 2b coincide when A is real. The maps $A \to A \cdot A^{-1}$ and $A \to A \cdot A^{-1T}$ are quite interesting in general and have been studied in [7].

Stronger than Theorems 2a and 2b are the following parallel results; the first was shown in [3], and the second is new. For an n-by-n matrix A, all of whose eigenvalues are real, denote the (algebraically) smallest of these eigenvalues by $\lambda_{\min}(A)$ and the largest by $\lambda_{\max}(A)$.

THEOREM 3a. *For* $A, B \in P_n$, *we have*

$$\lambda_{\min}(A \cdot B) \geqslant \lambda_{\min}(AB^T).$$

THEOREM 3b. *For* $A, B \in P_n$, *we have*

$$\lambda_{\min}(A \cdot B) \geqslant \lambda_{\min}(AB).$$

Again, Theorems 3a and 3b coincide if A and B are real (in fact, if A *or* B is real). But if A and B are complex, the eigenvalues of AB and AB^T can be quite different, and $\lambda_{\min}(AB)$ and $\lambda_{\min}(AB^T)$ can differ. For example, if

$$A = \begin{bmatrix} 2 & i \\ -i & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -2i \\ 2i & 2 \end{bmatrix},$$

then $\sigma(AB) = \{2 \pm \sqrt{2}\}$, $\sigma(AB^T) = \{6 \pm \sqrt{34}\}$, and $\sigma(A \cdot B) = \{4 \pm \sqrt{8}\}$; and we have $\lambda_{\min}(A \cdot B) = 4 - \sqrt{8} > \lambda_{\min}(AB) = 2 - \sqrt{2} > \lambda_{\min}(AB^T) = 6 - \sqrt{34}$.

It is an interesting tangential question whether there might be similar inequalities involving λ_{max}, and also what relationships there might be between AB and AB^T or between $A \cdot B$ and $A \cdot B^T$ in general.

It is clear that Theorem 3a implies Theorem 2a and that Theorem 3b implies Theorem 2b, while 3a and 2b or 3b are not directly comparable, for example. Both Theorems 3a and 3b imply Theorem 1, because the eigenvalues of a product of positive definite matrices are positive. Theorem 3a was first observed in [3], but as far as we know the distinct Theorem 3b has not previously been observed.

Our goal here is threefold:

(1) to exhibit the new result Theorem 3b;

(2) to present a way of dealing with Hadamard products which relates them to conventional matrix multiplication and yields strikingly simple proofs for Theorems 1, 2a and b, and 3a and b; and

(3) to give a converse for Theorem 2b

Also the proofs of Theorems 2a and 2b and those of 3a and 3b are essentially unified with a new lemma, which may be thought of as an analog to the fact that the Frobenius norm of a matrix dominates the root sum squared of the absolute values of the eigenvalues. Several proofs are known for Theorem 1, some of which are brief through use of the Kronecker product, but both published proofs of Theorem 2a and b (one of 2a and one of 2b) are somewhat elaborate. It is worth noting that proof techniques based upon the Kronecker product, which succeed nicely for Theorem 1, appear unable to handle the successively more subtle facts, Theorems 2a, b and 3a, b.

What is immediately striking about Theorem 2b is that the inequality may be rewritten as

$$A \cdot A^{-1} \geqslant AA^{-1},$$

so that Hadamard multiplication dominates conventional multiplication when the two multiplicands are functionally related (namely by the inversion function). Of course the left-hand side is positive definite by Theorem 1, but that it should dominate the usual product is remarkable. The point of our converse is that inversion is essentially unique in this regard.

We say that $F: P_n \to P_n$ is an *ordinary function* on P_n if, for $A \in P_n$ with unitary diagonalization

$$A = U^* \operatorname{diag}(\lambda_1, \dots, \lambda_n) U,$$

$F(A) = U^* \operatorname{diag}(f_1(\lambda_1, \dots, \lambda_n), \dots, f_n(\lambda_1, \dots, \lambda_n)) U$ for some given functions

$f_i: R_n^+ \to R^+$, $i=1,\dots,n$. Polynomials with positive coefficients, inversion, and exponentiation are examples of ordinary functions on P_n, but in each of these cases f_i depends only upon λ_i and all f_i are the same. The classical adjoint, $F(A)=(\det A)A^{-1}$, is an example of an ordinary function on P_n in which the f_i's depend upon more of the λ_j's and the f_i's are different: $f_i(\lambda_1,\dots,\lambda_n)=\prod_{j\neq i}\lambda_j$. Of course, the class of ordinary functions is very broad. Our motivation for this definition is that we want to consider a wide class of functions such that $A \in P_n$ implies $F(A)\in P_n$ and $AF(A)$ is Hermitian; for, these are the circumstances in which the issue of a converse to Theorem 2b is meaningful. The only aspect in which the ordinary functions on P_n are less general than this is that F is not allowed to depend upon the unitary matrix U. If it were, aside from the problem of definitional ambiguity, the definition of F could be modified for diagonal A, and there, since Hadamard and conventional multiplication coincide for diagonal matrices, $A\cdot F(A) \geqslant AF(A)$ trivially as long as $F(A)$ is diagonal.

Our converse to Theorem 2b is contained in the following.

THEOREM 4. *Let F be an ordinary function on P_n. Then*

$$A\cdot F(A) \geqslant AF(A) \qquad \text{for all} \quad A \in P_n$$

if and only if

for each $A \in P_N$, $F(A)$ is a positive scalar multiple of A^{-1}.

Facts such as Theorems 1 and 2 may be proven by first making the following observations, which hold for arbitrary n-by-n complex matrices A and B. For $x \in \mathbf{C}^n$, let D_x be the diagonal matrix whose ith diagonal entry is x_i, $i=1,\dots,n$, so that $D_x e = x$, in which e is the vector of 1's, as usual. As is easily verified, we also note that Hadamard multiplication commutes with conventional diagonal multiplication in the following sense: $D(A\cdot B)=DA\cdot B = A\cdot DB$ and $(A\cdot B)E = A\cdot BE = AE\cdot B$, whenever D and E are diagonal. We then have

$$\begin{aligned} x^*(A\cdot B)x &= e^T D_x^*(A\cdot B)D_x e \\ &= e^T(D_x^* A D_x \cdot B)e = \operatorname{Tr}\left(D_x^* A D_x B^T\right). \end{aligned} \qquad (*)$$

A proof of Theorem 1, then, just relies on the well-known observation that the product of a nonzero element of $\bar{P}_n$ and an element of P_n is diagonalizable and has nonnegative eigenvalues, which therefore cannot all be zero. If $A, B \in P_n$, then $D_x^*AD_x$ ($\in P_n$) is nonzero as long as $x \neq 0$, and $B^T \in P_n$; thus, $\mathrm{Tr}(D_x^*AD_xB^T) > 0$ and $x^*(A \cdot B)x > 0$, implying $A \cdot B \in P_n$.

Note that a fact about the Hadamard product of positive definite matrices has been related to a fact about the usual product of positive definite matrices.

A quite similar proof may be given for Theorem 2a and, with one additional observation, for Theorem 2b. This fact will also enable us to verify the new Theorem 3b.

LEMMA. *If C is a nonsingular n-by-n normal matrix, E is an n-by-n diagonal matrix, and $\|\cdot\|_F$ denotes the Frobenius matrix norm [$\|A\|_F^2 = Tr(A^*A)$], then*

$$\|C^{-1}EC^T\|_F \geq \|E\|_F.$$

Proof. Let $C = U^*DU$ be a unitary diagonalization of C. Then $\|C^{-1}EC^T\|_F = \|U^*D^{-1}UEU^TD\bar{U}\|_F = \|D^{-1}UEU^TD\|_F \geq \|UEU^T\|_F = \|E\|_F$. The second and last equalities are due to the fact that the Frobenius norm is unitarily invariant, and the inequality is due to the fact that UEU^T is symmetric and that $(1/|t|) + |t| \geq 2$ for all $0 \neq t \in \mathbf{C}$. ∎

The lemma will be applied to situations in which C is actually Hermitian, in which case $C^T = \bar{C}$. It should be noted that the inequality

$$\|C^{-1}E\bar{C}\|_F \geq \|E\|_F$$

holds also, even for general nonsingular C (the proof is the same after writing C in singular value form), while the inequality of the lemma does *not* hold for general nonsingular C: Let

$$C = \begin{bmatrix} 1 & -10 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}.$$

But this is not critical here.

Now, let $A \in P_n$, and $B = A^{-1T}$ (to prove Theorem 2a) or $B = A^{-1}$ (to prove Theorem 2b). Using (∗), the statement $A \cdot B \geq I$ is equivalent to

$$\mathrm{Tr}(D_x^*AD_xB^T) \geq \mathrm{Tr}(D_x^*D_x) = x^*x.$$

However,

$$\operatorname{Tr}(D_x^* A D_x B^T) = \operatorname{Tr}(B^{1/2T} D_x^* A^{1/2} A^{1/2} D_x B^{1/2T})$$
$$= \|A^{1/2} D_x B^{1/2T}\|_F^2 \geqslant \|D_x\|_F^2 = \operatorname{Tr}(D_x^* D_x).$$

The inequality follows from the lemma in case $B = A^{-1}$ (proving Theorem 2b), or from the more familiar fact that the square of the Frobenius norm dominates the sum of the squares of the absolute values of the eigenvalues in case $B = A^{-1T}$ (proving Theorem 2a). ■

Our proof of Theorem 3b is a refinement of the calculations used previously. It again uses the lemma in place of the classical inequality used in the proof of Theorem 3a in [3]. We have

$$\begin{aligned}
\lambda_{\min}(A\cdot B) &= \min_{x^*x=1} x^*(A\cdot B)x \\
&= \min_{\|D_x\|_F=1} \operatorname{Tr}(D_x^* A D_x B^T) \\
&= \min_{\|D_x\|_F=1} \|A^{1/2} D_x B^{1/2T}\|_F^2 \\
&= \min_{\|D_x\|_F=1} \|A^{1/2} B^{1/2} B^{-1/2} D_x B^{1/2T}\|_F^2 \\
&\geqslant \left\|(A^{1/2}B^{1/2})^{-1}\right\|_2^{-2} \min_{\|D_x\|_F=1} \|B^{-1/2} D_x B^{1/2T}\|_F \\
&\geqslant \|B^{-1/2}A^{-1/2}\|_2^{-2} = \left[\lambda_{\max}(B^{-1}A^{-1})\right]^{-1} \\
&= \lambda_{\min}(AB).
\end{aligned}$$

Here, $\|\cdot\|_2$ denotes the spectral norm, and the first inequality is the same as that used in [3]. The second inequality follows from the lemma, and each of the equalities is either a standard fact or an algebraic manipulation.

Of course Theorem 3a may be proved in much the same way, as in [3], by using the familiar fact that $\|B^{-1/2T} D_x B^{1/2T}\|_F \geqslant \|D_x\|_F$ in place of the lemma.

To prove Theorem 4 we need only establish the necessity of the asserted property of F. Because of the positive homogeneity of the inequality under study, the sufficiency follows from Theorem 2b. Let $f_1, \dots, f_n : R_n^+ \to R^+$ be the functions which induce the ordinary function F, and let $\lambda_1, \lambda_2, \dots, \lambda_n > 0$

be arbitrary. We seek to show that

$$\lambda_1 f_1(\lambda_1,\dots,\lambda_n) = \lambda_2 f_2(\lambda_1,\dots,\lambda_n) = \cdots = \lambda_n f_n(\lambda_1,\dots,\lambda_n).$$

Then, since these are the eigenvalues of $AF(A)$, it follows that

$$AF(A) = \lambda_i f_i(\lambda_1,\dots,\lambda_n) I$$

Thus, $F(A)$ would be a positive multiple of A^{-1}, as asserted by Theorem 4. Our strategy is to show that for an arbitrary pair of distinct indices k, j,

$$\lambda_k f_k(\lambda_1,\dots,\lambda_n) = \lambda_j f_j(\lambda_1,\dots,\lambda_n).$$

Without loss of generality (the same proof would work for any pair) we show this for $k=1,\ j=2$. Let

$$U_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad U_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

and

$$A_i = U_i^* \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} U_i, \qquad i = 1,2.$$

We consider

$$\hat{A}_i = A_i \oplus \operatorname{diag}(\lambda_3,\dots,\lambda_n), \qquad i = 1,2.$$

Since

$$\hat{A}_i \cdot F(\hat{A}_i) - \hat{A}_i F(\hat{A}_i) = [A_i \cdot F(A_i) - A_i F(A_i)] \oplus 0_{n-2}$$

it suffices to consider the implications of

$$A_i \cdot F(A_i) \geqslant A_i F(A_i), \qquad i = 1,2.$$

A calculation reveals that

$$A_1 = \frac{1}{2}\begin{bmatrix} \lambda_1 + \lambda_2 & \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 \end{bmatrix},$$

$$F(A_1) = \frac{1}{2}\begin{bmatrix} f_1(\lambda) + f_2(\lambda) & f_1(\lambda) - f_2(\lambda) \\ f_1(\lambda) - f_2(\lambda) & f_1(\lambda) + f_2(\lambda) \end{bmatrix},$$

and

$$A_1F(A_1)=\frac{1}{2}\begin{bmatrix}\lambda_1 f_1(\lambda)+\lambda_2 f_2(\lambda) & \lambda_1 f_1(\lambda)-\lambda_2 f_2(\lambda)\\ \lambda_1 f_1(\lambda)-\lambda_2 f_2(\lambda) & \lambda_1 f_1(\lambda)+\lambda_2 f_2(\lambda)\end{bmatrix},$$

in which we have used $f_i(\lambda)$ to denote $f_i(\lambda_1,\dots,\lambda_n)$, $i=1,2$. It follows that

$$\begin{aligned}&A_1\cdot F(A_1)-A_1F(A_1)\\ &=\frac{1}{4}\begin{bmatrix}f_1(\lambda)\lambda_2+\lambda_1 f_2(\lambda) & 3\lambda_2 f_2(\lambda)-\lambda_1 f_1(\lambda)\\ \quad-[\lambda_1 f_1(\lambda)+\lambda_2 f_2(\lambda)] & \quad-\lambda_1 f_2(\lambda)-\lambda_2 f_1(\lambda)\\ 3\lambda_2 f_2(\lambda)-\lambda_1 f_1(\lambda) & f_1(\lambda)\lambda_2+\lambda_1 f_2(\lambda)\\ \quad-\lambda_1 f_2(\lambda)-\lambda_2 f_1(\lambda) & \quad-[\lambda_1 f_1(\lambda)+\lambda_2 f_2(\lambda)]\end{bmatrix}.\end{aligned}$$

For $A_1\cdot F(A_1)-A_1F(A_1)$ to be positive semidefinite, it is necessary that

$$\lambda_2 f_2(\lambda)\geqslant\lambda_1 f_1(\lambda)$$

(because

$$(1,1)[A_1\cdot F(A_1)-A_1F(A_1)]\begin{pmatrix}1\\1\end{pmatrix}$$

must be nonnegative). Parallel calculations involving

$$A_2=\frac{1}{2}\begin{bmatrix}\lambda_1+\lambda_2 & \lambda_2-\lambda_1\\ \lambda_1-\lambda_2 & \lambda_1+\lambda_2\end{bmatrix}$$

reverse the roles of the first and second variables and functions to produce

$$\lambda_1 f_1(\lambda)\leqslant\lambda_2 f_2(\lambda).$$

We conclude that $\lambda_1 f_1(\lambda)=\lambda_2 f_2(\lambda)$ and that $\lambda_k f_k(\lambda)=\lambda_j f_j(\lambda)$ in general, completing the proof of Theorem 4.

If in addition to F being an ordinary function, we had also assumed that each f_i depends only upon λ_i, similar arguments would imply that each f_i is the same function (call it f) and that $\alpha f(\alpha)$ is constant for all $\alpha>0$. The stronger conclusion would then follow that F must have the more special form $F(A) = cA^{-1}$ for all $A \in P_n$ and some $c > 0$, *independent* of A. In the more general setting of an ordinary F, this strong of a conclusion may not be reached (c must be allowed to depend upon A, in essence). For example, $F(A) \equiv (\det A)A^{-1}$ is an ordinary function which satisfies the inequality of Theorem 4.

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