Integral Inequalities for Increasing Functions.
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Abstract. For numbers of increasing real functions \( f(x) \) with \( \int_{-1}^{+1} f(x) \, dx \geq 0 \) we give new integral inequalities. They generalize classical results. The proofs are short and simple being based on sequences.

1. Introduction. Let \( E \) be the set of all real functions \( f(x) \) defined and increasing for \(-1 \leq x \leq 1\). Let \( F \) be the set of members of \( E \) with

\[
0 \leq \int_{-1}^{1} f = \int_{x=-1}^{1} f(x) \, dx \tag{1}
\]

Also let \( G \) be the subset of \( F \) with equality in (1). Our main results are:

**THEOREM 1.** If \( f_1, \ldots, f_r \in E \) and

\[
0 \leq f_i(0) + \int_{0}^{1} f_i(x) \, dx \quad \text{for} \quad 1 \leq i \leq r \tag{2}
\]

then

\[
0 \leq \left[ \int_{0}^{1} f_1 \right] \cdots \left[ \int_{0}^{1} f_r \right] \leq \int_{0}^{1} f_1 \cdots f_r .
\]

Notice that for \( f_i \) defined and increasing for \( 0 \leq x \leq 1 \) the condition (2) simply requires that \( f_i \) can be extended to lie in \( F \).

**THEOREM 2.** If \( r \) is even and \( f_1, \ldots, f_r \in G \) then

\[
0 \leq \int_{-1}^{1} f_1 \cdots f_r .
\]
THEOREM 3. If \( f_1, \ldots, f_r, g_1, \ldots, g_s \in F \) and \( 0 \leq \theta \leq 1 \) then
\[
\left( \int_0^1 f_1 \ldots f_r \right) \frac{1}{\theta} \left( \int_0^1 g_1 \ldots g_s \right) \leq (1 - \theta) \left( \int_0^1 f_1 \ldots f_r g_1 \ldots g_s \right).
\]

THEOREM 4. If \( r, s \) are odd and \( f_1, \ldots, f_r, g_1, \ldots, g_s \in G \) then
\[
\left( \int_{-1}^1 f_1 \ldots f_r \right) \frac{1}{-1} \left( \int_{-1}^1 g_1 \ldots g_s \right) \leq \left( \int_{-1}^1 f_1 \ldots f_r g_1 \ldots g_s \right).
\]

These results will follow immediately from their analogues for sequences which we proceed to prove and discuss.

2. Finite increasing sequences. Let \( n \) be a fixed positive integer.

Abusing our notation we now let \( E \) be the set of all real sequences \( f(1) \leq \ldots \leq f(n) \). We let \( F \) be the members of \( E \) with
\[
0 \leq \sum f = f(1) + \ldots + f(n) \quad (3)
\]

Also we let \( G \) be the subset of \( F \) with equality in (3).

THEOREM 1'. If \( f_1, \ldots, f_r \in E \) and
\[
0 \leq (n - 1) f_1(1) + f_i(2) + f_i(3) + \ldots + f_i(n) \quad \text{for} \ 1 \leq i \leq r
\]

then
\[
0 \leq (n^{-1} \sum f_1) \ldots (n^{-1} \sum f_r) \leq n^{-1} \sum f_1 \ldots f_r.
\]

The familiar Chebychev type inequality ([3], 2.17) says that if \( f_1, \ldots, f_r \in E \) and are non-negative then for positive integers \( s \)
\[
\left( n^{-1} \sum f_1^s \right)^{1/s} \ldots \left( n^{-1} \sum f_r^s \right)^{1/s} \leq \left( n^{-1} \sum (f_1 \ldots f_r)^s \right)^{1/s}.
\]

Clearly this inequality follows immediately from Theorem 1'. It is more convenient to prove a slightly different form of Theorem 1' namely
THEOREM 1'. If \( f_1, \ldots, f_r \in F \) and \( t \) is an integer in \( \frac{1}{2n} \leq t \leq n \) then

\[
u_1 \ldots \nu_r \leq m^{-1} \sum f_1(x) \ldots f_r(x)\tag{4}
\]

where \( 0 \leq \nu_i = m^{-1} \sum f_i(x) \) for \( 1 \leq i \leq r \),

and \( m = n - t + 1 \) while summation is over \( t \leq x \leq n \).

Proof. We may assume that there is a smallest integer \( \lambda \) in

\( t-1 \leq p \leq n \) such that for each \( i \) in \( 1 \leq i \leq r \) we have

\( 0 < f_i(p+1) = \ldots = f_i(n) = g_i \) say. If \( t-1 = p \) then (4) holds

with equality. So assume \( t-1 < p \) and put \( q = n-p \) and \( h_i = f_i(p) \)

and \( k_i = (qg_i + h_i)/(q+1) \). Then for each \( i \) because \( f_i \in F \)

we have \( k_i \leq g_i \) and \( 0 < ph_i + qg_i \), so \( \left| h_i \right| \leq g_i \) so \( h_i \leq k_i \)

and \( 0 \leq k_i \). We change \( f_i \) to a new function \( f_i^* \) by changing

\( f_i(x) \) to \( k_i \) for \( p \leq x \leq n \). Then \( f_i^* \in F \) and has the same \( \nu_i \)

as \( f_i \). Further \( f_1^* \ldots f_r^* \leq f_1 \ldots f_r \) with summation over

\( t \leq x \leq n \), because

\[
(q + 1) k_i \leq (q \Pi g_i) + \Pi h_i \tag{5}
\]

with products over \( 1 \leq i \leq r \). The result (4) follows by repetition

of this process. It is easy to prove (5) by induction on \( r \).

If \( r \) is to be allowed to get large the condition \( \frac{1}{2n} \leq t \)

of Theorem 1' is necessary. To see this let all \( f_i \in G \) and be 1

for \( \frac{1}{2(n+1)} \leq x \leq n \) and constant elsewhere.

THEOREM 2'. If \( r \) is even and \( f_1, \ldots, f_r \in G \) then \( 0 \leq f_1 \ldots f_r \)

with summation over \( 1 \leq x \leq n \).
Proof. Split the sum at $\frac{1}{2}n$ and apply Theorem 1" to each half.

Inversion and change of sign for each $f_i$ shows there is no such result for $r$ odd.

THEOREM 3'. If $f_1, \ldots, f_r, g_1, \ldots, g_s \in F$ and $\frac{1}{2}n \leq t \leq n$ put

$$A = \Sigma f_1 \ldots f_r, \; B = \Sigma g_1 \ldots g_s, \; C = \Sigma f_1 \ldots f_r g_1 \ldots g_s$$

with summation over $t \leq x \leq n$ then $AB \leq (n - t + 1)C$.

Proof. We may assume $f_i(n) = g_j(n) = 1$ for all $i, j$. Then there will be a smallest integer $p$ in $t-1 \leq p \leq n$ such that $f_i(x) = g_j(x) = 1$ for $p < x \leq n$ and all $i, j$. If $t-1 = p$ the result holds with equality, so assume $t-1 < p$.

Now $-1 \leq f_i(p), g_j(p) \leq 1$ for all $i, j$. If say $f_i(p), f_2(p) < 0$ then we change $f_1, f_2$ into two new functions $f_1^*, f_2^*$ by changing $f_1(p), f_2(p)$ into $-f_1(p), -f_2(p)$ respectively. Clearly $f_1^*, f_2^* \in F$ and $A, B, C$ do not change. So we may assume $0 \leq f_2(p), \ldots, f_r(p)$ and that $c = f(p) < 1$ where $f$ now denotes $f_1$.

Put $q = n - p$ and $d = f_2(p) \ldots f_r(p)$ and $e = g_1(p) \ldots g_s(p)$ and $b = (q + cd)/(q + d)$. Notice that $0 \leq d \leq 1$ so $-1 \leq c \leq b$ and $0 \leq b$, and trivially $-1 \leq e \leq 1$. We change $f$ into a new function $f^*$ by changing $f(x)$ to $b$ for $p \leq x \leq n$. Let $A^*, B^*, C^*$ denote the corresponding new values of $A, B, C$. Now $f^*$ is increasing and the inequality $Ef \leq Ef^*$ is equivalent to $0 \leq q(1 - c)(1 - d)$ so $f^* \in F$. Observe that $A^* = A$ by definition
of b, and trivially $B^* = B$. Finally the inequality $C^* \leq C$ holds because it is equivalent to $qb + bde \leq q + cde$ which is $0 \leq qd(1 - c)(1 - e)$. 

If $b = 0$ then $f^* = 0$ and the result holds. If $0 < b$ we divide $f^*$ by $b$ and go back to the beginning of the proof. The theorem follows by repetition of this process.

**THEOREM 6'.** If $r, s$ are odd and $f_1, \ldots, f_r$, $g_1, \ldots, g_s \in G$ and $A, B, C$ are defined by (6) with summation over $1 \leq x \leq n$ then $AB \leq \frac{1}{2} n C$.

**Proof.** Suppose first that $n$ is even. We use (6) to define $A_1$, $B_1$, $C_1$ with summation over $1 \leq x \leq \frac{1}{2} n$ and $A_2$, $B_2$, $C_2$ with summation over $\frac{1}{2} n < x \leq n$. Thus $A = A_1 + A_2$ and similarly for $B, C$.

Now Theorem 3' says that $A_2B_2 \leq \frac{1}{2} n C_2$. If we multiply all $f_i, g_j$ by $-1$ it also says that $A_1B_1 \leq \frac{1}{2} n C_1$. Similarly from Theorem 1' we find that $A_1, B_1 \leq 0 \leq A_2, B_2$. It is now clear that $AB \leq \frac{1}{2} n C$.

This case $n$ even of this theorem yields Theorem 4 which in turn contains the case $n$ odd of this theorem.

We now give an example to show that the constant $\frac{1}{2} n$ in Theorem 4' is best possible. We let all $f_i$ be $-1, \ldots, -1, 0, \ldots, 0, p$ and all $g_j$ be $a, \ldots, a, 1, \ldots, 1$ with $a = -\left(\frac{1}{2} n - 1\right)/(\frac{1}{2} n + 1)$ then $A \sim p^r$ and $B \sim \frac{1}{2} n - 1$ while $C \sim p^r$. Examples of the form $-1, \ldots, -1, n-1$ and $-n+1, 1, \ldots, 1$ indicate that there are no other inequalities between $AB$ or $|A||B|$ and $C$ or $|C|$ with summation over $1 \leq x \leq n$. \n
DEFINITION. We say non-negative real numbers \( w(t), \ldots, w(n) \) are good weights if \( \frac{1}{n} \leq t \) and for all \( f_1, \ldots, f_r \in F \) we have

\[
0 \leq Ew f_1 \cdots f_r
\]

with summation over \( t \leq x \leq n \).

Thus good weights are related to Theorems 1, 1', 1''. We could not find weights for the other theorems.

Let \( H \) be the set of all \( f \in G \) of the form \(-p/q, \ldots, -p/q, 0, \ldots, 0, 1, \ldots, 1\) where \( q, n-p-q, p \) terms have the value \(-p/q, 0, 1\) respectively and the positive integers \( p, q \) have \( p+q \leq n \). It is easy to see that \( H \) is a basis for \( G \). If we adjoin the function \( 1, \ldots, 1 \) to \( H \) we get a basis for \( F \).

THEOREM 5. The non-negative reals \( w(t), \ldots, w(n) \) with \( \frac{1}{n} \leq t \) are good weights iff (7) holds whenever \( r = 1 \) and \( f_1 \in H \).

Proof. Necessity is obvious, so to show sufficiency let \( f_1, \ldots, f_r \in F \).

By linearity we may assume \( f_1, \ldots, f_r \in H \). There is a least \( p \) in \( t-1 \leq p \leq n \) such that \( f_1(x) = 1 \) for all \( i \) and \( p < x \leq n \).

If \( t-1 = p \) then (7) clearly holds, so assume \( t-1 < p \) and \( f_1(p) < 1 \).

Then by inspection of the functions in \( H \) we see that \( w(x)f_1(x) \leq w(x)f_1(x) \cdots f_r(x) \) for \( t \leq x \leq m \) and the theorem is proved.

3. Remarks on Lattices. The FKG and GKS inequalities of physics have many applications (see [1, 2, 4, 5]). It was trying to generalise them that led to this paper. Let \( L \) be the lattice of subsets of
finite set. Examples show that our above results do not generalise
to L. For \( \alpha, \lambda \in L \) let \( \sigma_\alpha(\lambda) \) be 1 if \( \alpha \subset \lambda \) but -1 otherwise. The case \( |\alpha| = 1 \) of these functions \( \sigma_\alpha \) is used in
physics. We do not allow \( |\alpha| = 0 \). Then it is easy to see that
\[ \sum_\alpha \leq 0 \leq \sum_\alpha \sigma_\beta, \]
where summation is over \( \lambda \in L \). We have proved
that \( \sum_\alpha \sigma_\beta \sigma_\gamma \) is \( >0 \) if \( 1 = |\alpha| < |\beta| \) and \( \alpha \neq \beta \) and \( \alpha \cup \beta \subset \gamma \). is \( =0 \) if
\( \alpha = \{1, 2\}, \beta = \{2, 3\}, \gamma = \{1, 3\} \), but is \( <0 \) otherwise.
Elementary arguments show that \( 0 \leq (-1)^r \sum_\alpha \sigma_\alpha_1 \ldots \sigma_\alpha_r \) if \( r = 4 \) and
\( |\alpha_r| = 2 \) or if \( r \leq 2^s-1 \) and \( s \leq |\alpha_1| \). We omit the proofs.

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