On Some Algebraic Problems in Connection with General Eigenvalue Algorithms

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ABSTRACT

Two real matrices A, B are S-congruent if there is a nonsingular upper triangular matrix R such that $A = R^TBR$. This congruence relation is studied in the set of all nonsingular symmetric and that of all skew-symmetric matrices. Invariants and systems of representation are given. The results are applied to the question of decomposability of a matrix in a product of an isometry and an upper triangular matrix, a problem crucial in eigenvalue algorithms.

INTRODUCTION

In studying the theoretical foundations of certain algorithms for the numerical determination of the spectrum of a matrix the following problem arises: For given real symmetric (skew-symmetric) nonsingular matrices A and B, determine if there exists an upper triangular matrix R such that

$$A = R^TBR.$$

This defines an equivalence relation, which we study here in some detail (Secs. 3, 4). It turns out that it is useful to investigate also the more general equivalence relation $\underset{T}{\sim}$, where $A \underset{T}{\sim} B$ means that there exists a lower triangular nonsingular matrix L and an upper triangular nonsingular matrix R such that $A = LBR$ (Sec. 2). In both cases we give invariants and representatives of these equivalence relations.

In Sec. 5 we apply some of the results to the question of decomposability of matrices in the form $G \cdot R$, where G is an isometry of a given bilinear form and R is an upper triangular matrix. This decomposability is crucial for certain numerical algorithms for the algebraic eigenvalue problem. It is shown that only in a few cases is the set of decomposable matrices not too small. These cases are studied separately at the end of Sec. 5.

1. NOTATION

In this paper all matrices are real. We denote the set of all $n\times m$ matrices by $\mathrm{M}_{n,m}$, the $n\times n$ matrices by M_n, the nonsingular $n\times n$ matrices by GL_n, the symmetric matrices in GL_n by S_n, the skew-symmetric matrices in GL_n by SK_n, the upper (lower) triangular matrices in GL_n by T_n^+ (T_n^-), the matrices in T_n^+ (T_n^-) with unit diagonal by $\mathrm{T}_{n,1}^+$ $(\mathrm{T}_{n,1}^-)$, the $n\times n$ permutation matrices by Π_n, the diagonal matrices in GL_n by D_n and those with diagonal ± 1 by D_n^1. For $A=(a_{ij})\in \mathrm{M}_{n,m}$ we denote by $A_{s,t}$ the submatrix consisting of the first s rows and the first t columns of A.

2. T-CONGRUENCE

Definition. Two matrices $A,B\in\mathrm{GL}_n$ are called *T*-congruent (triangularly congruent) if there exist $L\in\mathrm{T}_n^-$ and $R\in\mathrm{T}_n^+$ such that

$$A=LBR. \tag{1}$$

We write $A \underset{T}{\sim} B$.

We observe at once

$$A_{i,j}=L_{i,i}B_{i,j}R_{j,j},\qquad i,j=1,\dots,n, \tag{2}$$

and hence

$$A\underset{T}{\sim}B \quad\Rightarrow\quad \operatorname{rank}(A_{i,j})=\operatorname{rank}(B_{i,j}),\quad i,j=1,\dots,n. \tag{3}$$

In order to establish the reverse implication (Theorem 2) we have to prove

Theorem 1. *For $A\in\mathrm{M}_n$ there exist $L\in\mathrm{T}_{n,1}^-$, $P\in\Pi_n$ and an upper triangular matrix R such that*

$$A=LPR. \tag{4}$$

If $A \in \mathrm{GL}_n$, *then there exists exactly one* $P \in \Pi_n$ *such that* $A \underset{T}{\sim} P$.

REMARK. This result may be somewhere in the literature. It can be obtained easily from a result of Della Dora [1, II. V.8]; hence we give here only an indication of the proof.

Proof. Define recursively a permutation $\pi_1, \dots, \pi_n$ and matrices $A^{(r)} = (a_{ij}^{(r)})$, $r = 0, \dots, n$, by the following procedure: $A^{(0)} = A$, $I_1 = \{ j \in \{1, \dots, n\}, a_{j1}^{(0)} \neq 0\}$,

$$\pi_1 = \begin{cases} \min I_1 & \text{if } I_1 \neq \varnothing, \\ \text{any } j \in \{1, \dots, n\} & \text{otherwise.} \end{cases}$$

Eliminating the other nonzero elements in the first column by row operations always using row π_1 gives $A^{(1)}$.

If $\pi_1, \dots, \pi_{r-1}$ and $A^{(r-1)}$ are constructed, define

$$I_r = \{ j \in \{1, \dots, n\}, j \neq \pi_1, \dots, \pi_{r-1}, a_{jr}^{(r-1)} \neq 0\}$$

and

$$\pi_r = \begin{cases} \min I_r & \text{if } I_r \neq \varnothing, \\ \text{any } j \neq \pi_1, \dots, \pi_{r-1} & \text{otherwise.} \end{cases}$$

Eliminating all elements $a_{jr}^{(r-1)} \neq 0$, $j \neq \pi_1, \dots, \pi_r$ (for all these, $j > \pi_r$), by using row π_r only, we have $A^{(r)}$.

By construction $A^{(n)} = L^{-1}A$ for a suitable $L \in \mathrm{T}_{n,1}^-$ and $a_{ij}^{(n)} = 0$ for $i \neq \pi_1, \dots, \pi_j$, $j = 1, \dots, n$. Hence $A^{(n)} = PR$, where R is upper triangular and $P = (p_{ij}) = (\delta_{\pi_j i}) \in \Pi_n$. This shows (4). If $A \in \mathrm{GL}_n$, then $R \in \mathrm{T}_n^+$ and $A \underset{T}{\sim} P$. By (3) the ranks of $P_{i,j}$ are determined by A. But the ranks determine P uniquely as

$$\pi_i = \min\{ \nu : \operatorname{rank}(P_{\nu,i}) = \operatorname{rank}(P_{\nu,i-1}) + 1\} \tag{5}$$

for $i = 1, \dots, n$, with $\operatorname{rank}(P_{\nu,0}) = 0$ formally. ∎

THEOREM 2. *Let* $A, B \in \mathrm{GL}_n$. *Then*

$$A \underset{T}{\sim} B \quad \textit{iff} \quad \operatorname{rank}(A_{i,j}) = \operatorname{rank}(B_{i,j}), \quad i,j = 1, \dots, n.$$

Proof. Let $\operatorname{rank}(A_{i,j}) = \operatorname{rank}(B_{i,j})$. According to Theorem 1,

$$A \underset{T}{\sim} P \in \Pi_n, \qquad B \underset{T}{\sim} Q \in \Pi_n$$

and $\operatorname{rank}(P_{i,j}) = \operatorname{rank}(Q_{i,j})$. Equation (5) gives $P = Q$ and hence $A \underset{T}{\sim} B$. This together with (3) proves the theorem. ∎

By embedding matrices of $\mathrm{M}_{m,n}$ in quadratic matrices and using Theorem 1, similar results can be obtained for rectangular matrices.

THEOREM 3. *For $A \in \mathrm{GL}_n$ there exist $R \in \mathrm{T}^+_{n,1}$, $L \in \mathrm{T}^-_{n,1}$, $P \in \Pi_n$, $D \in \mathrm{D}_n$ such that*

$$A = LPDR. \tag{6}$$

P and D are uniquely determined.

Proof. In view of Theorem 1, only the uniqueness of D has to be shown. Let $A = L_1 PDR_1 = L_2 P\Delta R_2$, where $\Delta \in D_n$; then

$$PD = LP\Delta R, \qquad L = L_1^{-1} L_2 \in \mathrm{T}^-_{n,1}, \quad R = R_2 R_1^{-1} \in \mathrm{T}^+_{n,1},$$

or $P^T L^{-1} PD = \Delta R$. Comparison of the elements in position (i,i) gives $d_i = \Delta_i$. ∎

Because of the importance of the ranks of $A_{i,j}$, the following result is of some interest.

THEOREM 4. *For mn given integers $r_{i,j}$ $(i = 1,\dots,n,\ j = 1,\dots,m)$ the following are equivalent*:

(1) *There exists $A \in \mathrm{M}_{n,m}$ such that*

$$\operatorname{rank}(A_{i,j}) = r_{i,j}, \qquad i = 1,\dots,n, \quad j = 1,\dots,m;$$

(2)

(i) $0 \leqslant r_{i,j} \leqslant \min(i,j)$, $i = 1,\dots,n$, $j = 1,\dots,m$,
(ii) $r_{i,j} \leqslant r_{i,j+1} \leqslant r_{i,j} + 1$, $i = 1,\dots,n$, $j = 1,\dots,m-1$,
(iii) $r_{i,j} \leqslant r_{i+1,j} \leqslant r_{i,j} + 1$, $i = 1,\dots,n-1$, $j = 1,\dots,m$,
(iv) $r_{i,j} = r_{i,j+1} \Rightarrow r_{k,j} = r_{k,j+1}$ *for* $k \leqslant i$, $i = 1,\dots,n$, $j = 1,\dots,m-1$,
(v) $r_{i,j} = r_{i+1,j} \Rightarrow r_{i,k} = r_{i+1,k}$ *for* $k \leqslant j$, $i = 1,\dots,n-1$, $j = 1,\dots,m$.

Proof. (1)⇒(2) is obvious. To show the reverse implication we assume that $r_{i,j}$ with (2) are given.

Let $r_{k,0}=0$, $k=1,\dots,n$. We define

$$I_j=\{k: r_{k,j}=1+r_{k,j-1}\},$$

$$s(j)=\begin{cases}\min I_j & \text{if } I_j\neq\varnothing\\ n+1 & \text{otherwise}\end{cases},\qquad j=1,\dots,m,$$

and

$$A=(a_{ij})=(\delta_{is(j)}),\qquad i=1,\dots,n,\quad j=1,\dots,m.$$

Then it is not difficult to show that A satisfies (1). ■

Let us remark that the inequalities in (2) are not independent. In fact we need only $r_{k,1}\leqslant 1$, $k=1,\dots,n$, (ii), (iii), (iv) to prove that the matrix A given in the proof of Theorem 4 satisfies (1). Hence the remaining inequalities follow.

3. THE SYMMETRIC CASE

Definition. Two matrices $A,B\in \mathrm{S}_n$ are called S-congruent (symmetrically triangularly congruent) if there exists $R\in \mathrm{T}_n^+$ such that

$$A=R^TBR. \tag{7}$$

We use the notation $A \underset{S}{\sim} B$.

We shall establish results similar to those referring to T-congruence. Let

$$\mathcal{S}_n=\{PD: P\in\Pi_n,\ P=P^T,\ D=\operatorname{diag}(d_i)\in D_n^1,\ d_i=-1\Rightarrow p_{ii}=1\}. \tag{8}$$

$\mathcal{S}_n$ can be described as the set of all matrices which originate from symmetric permutation matrices by eventually replacing 1's in the diagonal by -1's.

The following is a result analogous to Theorems 1 and 3.

Theorem 5. *For any $A\in \mathrm{S}_n$ there exists $R\in \mathrm{T}_n^+$ and a unique $Q\in\mathcal{S}_n$ such that*

$$A=R^TQR. \tag{9}$$

Proof. As a first step we prove the following: Let $B=(b_{ij})\in \mathbb{S}_n$. Denote by $I(B)$ the maximal set of indices i such that for $i\in I(B)$ $b_{ik}\neq 0$ for exactly one $k\in\{1,\dots,n\}$ and this k is in $I(B)$. If $I(B)\neq\{1,\dots,n\}$, then it is possible to find $R\in \mathrm{T}^+_{n,1}$ such that

$$I(R^TBR)\underset{\neq}{\supset} I(B).$$

Let $i=\min\{k:k\notin I(B)\}$, $j=\min\{k:b_{ki}\neq 0\}$. Then $i,j\notin I(B)$. If $b_{ki}\neq 0$, $k>j$, eliminate b_{ki} by subtracting b_{ki}/b_{ji} times the ith row from the kth row. In the same way eliminate b_{ik} by a column operation. Hence we get $R_1\in \mathrm{T}^+_{n,1}$ such that $\tilde{B}=R_1^TBR_1$ has in addition only one nonzero element in row and column i. If $i=j$, then $I(B)\cup\{i\}\subset I(\tilde{B})$ and we are ready.

If $i\neq j$, one has in addition to eliminate the nonzero elements $\tilde{b}_{jk}$ and $\tilde{b}_{kj}$ $(k>i)$ by a similar procedure. Only the elimination of $\tilde{b}_{jj}$ has to be modified. To ensure the symmetry of the result, subtract $\frac{1}{2}\tilde{b}_{jj}/\tilde{b}_{ij}$ times the ith column from the jth column and similarly for the rows. Now for the resulting matrix, which is of the form R^TBR, $R\in \mathrm{T}^+_{n,1}$, one has $I(B)\cup\{i,j\}\subset I(R^TBR)$. Starting with $B=A$ we get after finitely many steps $\tilde{R}\in \mathrm{T}^+_{n,1}$ such that $\tilde{Q}=\tilde{R}^TA\tilde{R}$ satisfies $I(\tilde{Q})=\{1,\dots,n\}$, i.e., in each row and column of $\tilde{Q}$ there is exactly one nonzero element. We can find $D_1\in D_n$ such that in $D_1\tilde{Q}D_1=Q$ the nondiagonal nonzero elements are 1, while the diagonal nonzero elements are $+1$ or -1, according to the sign of this element in $\tilde{Q}$. Hence $Q\in \mathbb{S}_n$. Setting $(\tilde{R}D_1)^{-1}=R$, (9) is established.

Writing $Q=PD$ as in (8), we see from Theorem 1 that P is uniquely determined.

If $A=R^TPDR=S^TP\Delta S$, $R,S\in \mathrm{T}^+_n$, $PD,P\Delta\in \mathbb{S}_n$, then

$$PD=\tilde{R}^T\tilde{D}P\Delta\tilde{D}\tilde{R},$$

where $\tilde{R}\in \mathrm{T}^+_{n,1}$, $\tilde{D}\tilde{R}=SR^{-1}$. Theorem 3 gives

$$PD=\tilde{D}P\Delta\tilde{D},$$

which implies

$$p_{ii}\left(d_i-\Delta_i\tilde{d}_i^2\right)=0.$$

Hence for $p_{ii}=1$ we have $d_i=\Delta_i$, while for $p_{ii}=0$ the definition of $\mathbb{S}_n$ gives $d_i=\Delta_i=1$. This shows $D=\Delta$. ■

The description of the invariants of $\underset{S}{\sim}$ is not so easy. Obviously the ranks of $A_{i,j}$ are invariant, and likewise the numbers $\operatorname{sgn}\det(A_{i,i})$, $i=1,\dots,n$, but the two matrices $A_1, A_2 \in \mathbb{S}_4$

$$A_1 \doteq \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

which by Theorem 5 are not S-congruent, show that in general these numbers don't tell the whole story.

A complete characterization is given in Theorem 6.

We need some preparations. Let $B^+ \in \mathrm{M}_n$ denote the Moore-Penrose inverse of $B \in \mathrm{M}_n$, i.e., the matrix uniquely determined by

$$BB^+B = B, \tag{10.i}$$

$$B^+BB^+ = B^+, \tag{10.ii}$$

$$BB^+ = (BB^+)^T, \tag{10.iii}$$

$$B^+B = (B^+B)^T. \tag{10.iv}$$

(e.g. [8]). The range of B is denoted by $R(B)$, i.e.,

$$R(B) = \{b \in R^n : \exists x \in R^n, Bx = b\}$$

For $B \in \mathrm{M}_n$ $(n \geqslant 2)$ symmetric we consider the following partition:

$$B = \begin{pmatrix} B_{n-1} & b_{n-1} \\ b_{n-1}^T & b_{nn} \end{pmatrix}, \tag{11}$$

where $B_{n-1} \in \mathrm{M}_{n-1}$, $b_{n-1} \in \mathrm{M}_{n-1,1}$, $b_{nn} \in R$, and define

$$d(B) = b_{nn} - b_{n-1}^T B_{n-1}^+ b_{n-1}. \tag{12}$$

For later reference we state the well-known fact

$$d(B) = \frac{\det B}{\det B_{n-1}} \qquad \text{if} \quad \det B_{n-1} \neq 0, \tag{13}$$

which is also implied by (15).

LEMMA. *For $B \in M_n$ symmetric, the following are equivalent:*

(1) $\operatorname{rank}(B) = \operatorname{rank}(B_{n-1}) + 1$
(2) $b_{n-1} \in R(B_{n-1})$, $d(B) \neq 0$.

In this case

$$B \underset{S}{\sim} \begin{pmatrix} B_{n-1} & 0 \\ 0 & d(B) \end{pmatrix}. \tag{14}$$

If under the conditions (1) *or* (2) $B \underset{S}{\sim} C$ *for some symmetric* $C \in M_n$, *then*

$$\operatorname{sgn} d(B) = \operatorname{sgn} d(C).$$

Proof. We remark that if $B_{n-1}x = b_{n-1}$, then $x^T B_{n-1} x = b_{n-1}^T B_{n-1}^+ b_{n-1}$ and

$$B = \begin{pmatrix} I & 0 \\ x^T & 1 \end{pmatrix} \begin{pmatrix} B_{n-1} & 0 \\ 0 & d(B) \end{pmatrix} \begin{pmatrix} I & x \\ 0 & 1 \end{pmatrix}. \tag{15}$$

Hence (2) implies (1). If $b_{n-1} \notin R(B_{n-1})$, then

$$\operatorname{rank}(B_{n-1} | b_{n-1}) = 1 + \operatorname{rank}(B_{n-1}),$$

and if (1) is satisfied, the last row of B is a linear combination of the first $n-1$ rows; in particular

$$b_{n-1} \in R\left(B_{n-1}^T\right) = R(B_{n-1}).$$

This shows (1) $\Rightarrow$ $b_{n-1} \in R(B_{n-1})$ and (15) gives $d(B) \neq 0$. Let $B \underset{S}{\sim} C$. In view of (15) we assume $b_{n-1} = 0$, $d(B) = b_{nn}$. There exists

$$R = \begin{pmatrix} R_1 & r \\ 0 & \rho \end{pmatrix}$$

such that $C = R^T B R$. From

$$C = \begin{pmatrix} R_1^T B_{n-1} R_1 & R_1^T B_{n-1} r \\ r^T B_{n-1} R_1 & r^T B_{n-1} r + \rho^2 b_{nn} \end{pmatrix}$$

we get

$$d(C) = r^T B_{n-1} r + \rho^2 b_{nn} - r^T B_{n-1} R_1 \left(R_1^T B_{n-1} R_1\right)^+ R_1^T B_{n-1} r$$
$$= \rho^2 b_{nn},$$

the last equation following from (10.i) and the nonsingularity of R_1. ■

For $A \in \mathrm{S}_n$ define

$$d_1(A) = a_{11}, \qquad d_i(A) = d(A_{i,i}), \quad i = 2, \ldots, n. \tag{16}$$

THEOREM 6. *For $A, B \in \mathrm{S}_n$ the following are equivalent:*

(1) $A \underset{\mathrm{S}}{\sim} B$;

(2) rank $(A_{i,j})$ = rank $(B_{i,j})$, $i, j = 1, \ldots, n$. *If* $i = 1$ *or* rank $(A_{i,i}) = 1 +$ rank $(A_{i-1,i-1})$, *then*

$$\operatorname{sgn} d_i(A) = \operatorname{sgn} d_i(B).$$

Proof. (1)⇒(2) follows from the preceding lemma and Theorem 1. The case $i = 1$ is obvious.

(2)⇒(1): We may assume $A, B \in \mathfrak{S}_n$, $A = PD$, $B = \tilde{P}\tilde{D}$. The rank conditions give $P = \tilde{P}$. We show that if $D = \operatorname{diag}(d_i)$, $\tilde{D} = \operatorname{diag}(\tilde{d}_i)$, then $d_i = \tilde{d}_i$. If $p_{ii} = 0$, then $d_i = \tilde{d}_i = 1$ according to the definition of $\mathfrak{S}_n$. If $p_{11} = 1$, then $d_1 = \operatorname{sgn} d_1(A) = \operatorname{sgn} d_1(B) = \tilde{d}_1$. If $i \geq 2$ and $p_{ii} = 1$, then $1 + \operatorname{rank}(A_{i-1,i-1}) = \operatorname{rank}(A_{i,i})$. Here obviously $d_i = d_i(A)$, $\tilde{d}_i = d_i(B)$. Hence $d_i = \tilde{d}_i$ for all i. ■

COROLLARY 6. *Let $A \in \mathrm{S}_n$ such that* $\det(A_{i,i}) \neq 0$, $i = 1, \ldots, n$. *Then for $B \in \mathrm{S}_n$ the following are equivalent:*

$$A \underset{\mathrm{S}}{\sim} B, \tag{17}$$

$$\operatorname{sgn}\det(A_{i,i}) = \operatorname{sgn}\det(B_{i,i}), \qquad i = 1, \ldots, n. \tag{18}$$

Proof. (17)⇒(18) follows by (2).

If (18) is satisfied, then $\det(B_{i,i}) \neq 0$, $i = 1, \ldots, n$, and hence $\operatorname{rank}(A_{i,j}) =$

$\operatorname{rank}(B_{i,j}) = \min(i,j)$. By (13) $\operatorname{sgn} d_i(A) = \operatorname{sgn} d_i(B)$ for $i = 2,\ldots,n$, and the same for $i = 1$ by (18). Theorem 6 shows $A \underset{S}{\sim} B$. ■

This corollary can also be easily obtained by using well-known results on triangular decompositions (e.g. [9]).

4. THE SKEW-SYMMETRIC CASE

DEFINITION. $A, B \in \mathrm{SK}_n$ are called S-congruent if there exists $R \in \mathrm{T}_n^+$ such that

$$A = R^T B R.$$

Let $\mathfrak{T}_n$ denote the set of all matrices M such that M contains exactly one element ± 1 in each row and column, the rest zeros, $M = -M^T$ and $m_{ij} > 0$ for $i < j$. $\mathfrak{T}_n$ consists of all matrices which originate from symmetric permutation matrices with zero diagonal by setting the 1's below the diagonal equal to -1.

THEOREM 7. *For any $A \in \mathrm{SK}_n$ there exists $R \in \mathrm{T}_n^+$ and a unique $\Pi \in \mathfrak{T}_n$ such that*

$$A = R^T \Pi R.$$

For $A, B \in \mathrm{SK}_n$ the following are equivalent:

(1) $A \underset{S}{\sim} B$;
(2) $\operatorname{rank}(A_{i,j}) = \operatorname{rank}(B_{i,j})$, $i,j = 1,\ldots,n$.

Proof. The first part is proved as in Theorem 5 by symmetric elimination. It is easy to see that Π is uniquely determined by the numbers $\operatorname{rank}(A_{i,j})$ [see (3) and (5)]. This proves also (2)⇒(1). (1)⇒(2) follows from (3). ■

Let us remark that $\mathrm{SK}_n \neq \varnothing$ implies $n = 2k$ even.

COROLLARY 7. *Let $A \in \mathrm{SK}_n$, $\det A_{2i,2i} \neq 0$ $(i = 1,\ldots,k)$, and $B \in \mathrm{SK}_n$. Then*

$$A \underset{S}{\sim} B \quad \Leftrightarrow \quad \det B_{2i,2i} \neq 0, \quad i = 1,\ldots,k. \tag{19}$$

Proof. It is easy to prove by induction on k that the only matrix $\Pi \in \mathfrak{T}_n$ with $\det \Pi_{2i,2i} \neq 0$, $i=1,\ldots,k$, is given by

$$J = \operatorname{diag}(J_i)_{i=1,\ldots,k}, \qquad J_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Hence for a matrix $C \in \mathrm{SK}_n$ we have

$$C \underset{S}{\sim} J \quad \Leftrightarrow \quad \det C_{2i,2i} \neq 0, \quad i=1,\ldots,k.$$

This proves the corollary. ■

5. APPLICATION TO DECOMPOSABILITY

We study now the decomposition of a nonsingular real matrix A into a product $G \cdot R$ of a matrix G in a fixed given group $\mathcal{G}$ and an upper triangular matrix $R \in \mathrm{T}_n^+$:

$$A = G \cdot R. \tag{20}$$

This decomposition plays a certain role in solving linear systems and eigenvalue problems: If G is easily invertible,

$$Ax = r$$

is split into two easily solvable systems,

$$Gy = r,$$

$$Rx = y.$$

On the other hand, Della Dora has shown in [1] and [2] that for big enough $\mathcal{G} \cdot \mathrm{T}_n^+$ the algorithm

$$A_0 = A,$$

$$\left.\begin{aligned} A_i &= G_i R_i, \\ A_{i+1} &= R_i G_i, \end{aligned}\right\} \qquad i = 0, 1, \ldots,$$

under certain assumptions yields a sequence $\{A_i\}$ converging essentially to an upper triangular matrix the diagonal of which consists of the eigenvalues of A. We consider the case that $\mathcal{G}$ is the group of isometries of a symmetric or skew-symmetric bilinear form.

Let J be a nonsingular real symmetric (skew-symmetric) matrix. Then

$$(x,y)_J = x^T J y \tag{21}$$

describes a symmetric (alternating) nonsingular bilinear form (see e.g. [4]), and

$$G_J = \{ G \in \mathrm{M}_n : G^T J G = J \} \tag{22}$$

is the group of all isometries:

$$G \in G_J \quad \Leftrightarrow \quad (x,y)_J = (Gx, Gy)_J \quad \text{for all } x, y \in R^n. \tag{23}$$

In addition define

$$M_J = \{ GR : G \in G_J,\, R \in \mathrm{T}_n^+ \} = G_J \cdot \mathrm{T}_n^+ \tag{24}$$

and observe

$$A \in M_J \quad \Leftrightarrow \quad A^T J A \underset{S}{\sim} J. \tag{25}$$

Indeed from $A \in \mathrm{M}_J$ one has $A^T J A = R^T G^T J G R = R^T J R$, i.e., $A^T J A \underset{S}{\sim} J$. On the other hand $A^T J A \underset{S}{\sim} J$ means the existence of $R \in \mathrm{T}_n^+$ with $A^T J A = R^T J R$, whence $(AR^{-1})^T J A R^{-1} = J$ or $AR^{-1} \in G_J$.

We want to study M_J. Let us remark that we need only consider the cases $J \in \mathbb{S}_n$ and $J \in \mathfrak{T}_n$. This is a consequence of the following relation: For $R \in \mathrm{T}_n^+$

$$M_{R^T J R} = R^{-1} M_J R. \tag{26}$$

Proof of (26).

$$\begin{aligned} A \in M_{R^T J R} \quad &\Leftrightarrow \quad A^T R^T J R A = R_1^T R^T J R R_1 \\ &\Leftrightarrow \quad (RAR^{-1})^T J R A R^{-1} \\ &= (RR_1R^{-1})^T J (RR_1R^{-1}) \\ &\Leftrightarrow \quad RAR^{-1} \in M_J. \end{aligned}$$

The next theorem shows that in most cases M_J has Lebesgue measure zero in M_n, so that these decompositions are not suited for numerical computations.

THEOREM 8. *Let J satisfy one of the following conditions:*

(i) $J \in \mathrm{S}_n$, $\det J_{i,i} = 0$ *for some* $1 \leqslant i \leqslant n-1$.
(ii) $J \in \mathrm{SK}_n$, $\det J_{i,i} = 0$ *for some* $i = 2k$, $1 \leqslant 2k \leqslant n-1$.

Then M_J is of Lebesgue measure zero in M_n.

Proof. In case (i) it follows from Theorem 6 and in case (ii) from Theorem 7 and (25) that $A \in M_J$ satisfies

$$f(A) = \det(A^T JA)_{ii} = 0. \tag{27}$$

Now f is a polynomial, homogeneous of degree $2i$, in the unknowns a_{jl}. It is known (see [3], p. 83 and p. 84) that there exists $K \in \mathrm{GL}_n$ such that

$$K^T JK = \operatorname{diag}(1, \ldots, 1, -1, \ldots, -1) \qquad \text{in case (i)} \tag{28}$$

and

$$K^T JK = \operatorname{diag}(J_1, \ldots, J_1), \quad J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{in case (ii)}. \tag{29}$$

In any case $f(K) = \pm 1$. This shows that M_J is contained in the algebraic set $\{A \in \mathrm{M}_n, f(A) = 0\}$, which is of Lebesgue measure zero, as $f \neq 0$.

We must still study

$$J = \operatorname{diag}(j_i), \qquad j_i = \pm 1, \tag{30}$$

in the symmetric case, and

$$J = \operatorname{diag}(J_i), \qquad i = 1, \ldots, k, \quad J_i = J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad n = 2k, \tag{31}$$

in the skew-symmetric case, as these are the only matrices in $\mathcal{S}_n$ and $\mathcal{T}_n$ not satisfying the conditions of Theorem 8. This is done in the following theorems.

For given n and p $(1 \leqslant p \leqslant n)$, consider the set I_p of all sequences $(j_1, \ldots, j_n)$, $j_i = \pm 1$, p of the j_i's being $+1$. For any $j = (j_1, \ldots, j_n) \in I_p$ let

$$J_j = \operatorname{diag}(j_1, \ldots, j_n).$$

We remark that there are exactly $p!(n-p)!$ permutations Π such that

$$J_j = \Pi^T J_j \Pi$$

and that there are $\binom{n}{p}$ elements in I_p.

THEOREM 9. *Let $J = J_j$, $j \in I_p$, $A \in \mathrm{GL}_n$. Then the following are equivalent*:

(1) $\det[(A^T JA)_{i,i}] \neq 0$, $i = 1, \ldots, n$.
(2) *There exist* $S \in G_J$, $P \in \Pi_n$, $R \in \mathrm{T}_n^+$, *such that*

$$A = SPR.$$

In (2) *the matrices SP and R are uniquely determined, if in addition the diagonal of R is chosen to be positive.*

Proof. (1)⇒(2) is attributed to Graev [1]; we give a proof for completeness. According to Corollary 6, $A^T JA \underset{S}{\sim} \tilde{J}$, where $\tilde{J} \in \mathrm{D}_n^1$. As J is congruent to $\tilde{J}$, $\tilde{J} \in I_p$, we have $\tilde{J} = P^T JP$ for a suitable $P \in \Pi_n$. Therefore there exists $R \in \mathrm{T}_n^+$ with $A^T JA = R^T P^T JPR$. This shows $A(PR)^{-1} \in G_J$.

(2)⇒(1): From $A = SPR$ one has $A^T JA = R^T P^T S^T JSPR = R^T P^T JPR$, i.e., $A^T JA \underset{S}{\sim} P^T JP$, and hence (1).

Let us assume $A = SPR = S'P'R'$. Obviously $A^T JA \underset{S}{\sim} P^T JP \in \mathcal{S}_n$ and $A^T JA \underset{S}{\sim} P'^T JP' \in \mathcal{S}_n$, which implies $P^T JP = P'^T JP'$. Hence the matrix $C = (S'P')^{-1} SP$, which by assumption is equal to $R'R^{-1}$, is in G_J. On the other hand $C \in \mathrm{T}_n^+$, and we see from $C^T J = JC^{-1}$ and the positivity of the diagonal of C that $C = I$. ■

We remark the obvious fact (in view of corollary 7)

$$A \in M_J \quad \Leftrightarrow \quad \operatorname{sgn} \det(A^T JA)_{ii} = j_1 \cdots j_i, \quad i = 1, \ldots, n.$$

Hence M_J is open, but not dense in GL_n.

For fixed J and for all $P \in \Pi_n$ let

$$N_P = \{ SPR : S \in G_J, R \in \mathrm{T}_n^+ \}$$

Then

$$N_I = M_J, \qquad \overline{\bigcup_P N_P} = M_n$$

and we have the following

Theorem 10. *The following are equivalent for $P, Q \in \Pi_n$:*

(1) $N_P = N_Q$,
(2) $N_P \cap N_Q \neq \varnothing$,
(3) $PQ^T \in G_J$.

Proof. (3)⇒(1): follows from $SPR = S(PQ^T)QR$.

(1)⇒(2): Trivial.

(2)⇒(3): If $A = S_1PR_1 = S_2QR_2$, then $A^TJA = R_1^TP^TJPR_1 = R_2^TQ^TJQR_2$ or $P^TJPR = R^{-T}Q^TJQ$, $R = R_1R_2^{-1}$. This shows $R = \mathrm{diag}(\pm 1)$, $P^TJP = Q^TJQ$, and $PQ^T \in G_J$. ■

We see that except in the case $J = I$ (i.e., $p = n$), M_J is not dense, but that we have $\binom{n}{p}$ different open sets of matrices N_P, the union of which is dense in M_n.

From Theorem 9 we find that for given P, Q the transformation

$$SPR \in N(P) \rightarrow SQR \in N(Q)$$

is one-to-one if we restrict ourselves to R triangular with positive diagonal. In this sense the different N_P's are of the same "size."

As another consequence, G_J for $J \in I_p$, $p < n$, does not give a matrix decomposition suitable for eigenvalue algorithms. Nevertheless there exists a modification based on this decomposition, the so-called HR algorithm (see [5], [6], [7], [10]).

In the skew-symmetric case things are simpler. It remains to study the case (31) ($2k = n$):

$$J = I_k \times J_1,$$

where I_k is the k-dimensional unit matrix,

$$J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and $\times$ denotes the tensor product [3, p. 8].

Theorem 11. *For $A \in \mathrm{GL}_n$ the following are equivalent:*

(1) $A \in M_J$, *i.e.*, $A = S \cdot R$, $S \in G_J$, $R \in \mathrm{T}_n^+$;
(2) $\det[(A^TJA)_{i,i}] \neq 0$, $i = 2, 4, \ldots, 2n$.

Proof. (1)⇒(2): As $A^TJA \underset{S}{\sim} J$ and $\det J_{2i,2i} \neq 0$, Corollary 7 yields (2).

On the other hand, if (2) is satisfied, Corollary 7 gives $A^TJA \underset{S}{\sim} J$ and $A \in M_J$. ■

REMARK. Theorem 11 implies that M_J is big enough, indeed $\overline{M_J} = M_n$. This can be found in [2] already, but not the characterization (2). In [2] some algorithms for the actual computation of the decomposition are given.

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