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Another Note on a Theorem of Minc on Irreducible Nonnegative Matrices

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A short proof is given for a part of a theorem of Minc.

Minc [1] proved the following.

THEOREM *Suppose A is a nonnegative $n \times n$ matrix and for some permutation matrix P*

$$A = P \begin{bmatrix} 0 & A_1 & 0 & \dots & 0 \\ 0 & 0 & A_2 & \dots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & & & 0 & A_{h-1} \\ A_h & 0 & \dots & 0 & 0 \end{bmatrix} P^T \tag{1}$$

where the zero blocks on the diagonal are square. Then A is irreducible if and only if A has no zero rows and columns and $B = A_1 A_2 \dots A_h$ is irreducible.

Pullman [2] gave a short proof for the if-part using the following fact:

Define the relation $\ll$ for nonnegative n-tuples by writing $x \ll y$ iff $y_i > 0$ whenever $x_i > 0$ for $1 \leqslant i \leqslant n$. Then a nonnegative matrix C is irreducible iff $Cx \ll x$, $x \geqslant 0$, $x \neq 0$ implies $x > 0$.

Here we note that also the "only if"-part can be proved by using the same device. Assume $Bx_1 \ll x_1$, $x_1 \geqslant 0$, $x_1 \neq 0$. Define iteratively nonnegative vectors $x_h, \ldots, x_2$ by

$$A_h x_1 = x_h \tag{2}$$

$$A_r x_{r+1} = x_r \qquad r = h-1, \ldots, 2 \tag{3}$$

Then $Bx_1 \leqslant x_1$ gives

$$A_1 x_2 \leqslant x_1 \tag{4}$$

(2), (3), (4) imply $P^T APx \leqslant x$, whence $APx \leqslant Px$, for the nonnegative vector $x = (x_1, x_2, \ldots, x_h)^T$, $x \neq 0$. A being irreducible gives $Px > 0$, in particular $x_1 > 0$. Hence B is irreducible.

References

[1] H. Minc, The structure of irreducible matrices, *Linear and Multilinear Algebra*, **2** (1974), 85–90.

[2] N. J. Pullman, A note on a theorem of Minc on irreducible nonnegative matrices, *Linear and Multilinear Algebra*, **2** (1974), 335–336.