Inverse Iteration for Calculating the Spectral Radius of a Non-Negative Irreducible Matrix

Ludwig Elsner

Institut für Angewandte Mathematik der Universität Erlangen-Nürnberg
8520 Erlangen
Martensstrasse 1, Germany

Submitted by Hans Schneider

ABSTRACT

Noda established the superlinear convergence of an inverse iteration procedure for calculating the spectral radius and the associated positive eigenvector of a non-negative irreducible matrix. Here a new proof is given, based completely on the underlying order structure. The main tool is Hopf's inequality. It is shown that the convergence is quadratic.

1. INTRODUCTION

Throughout this paper $A$ will denote a non-negative irreducible $N \times N$ matrix with spectral radius $\rho$ and associated positive eigenvector $p$.

In [5], Noda established the convergence of an inverse iteration procedure for the determination of $\rho$ and $p$. He also showed that the convergence is superlinear. Here we shall prove that it is at least quadratic.

This is an easy by-product of our proof of convergence, which uses only the underlying order structure and not (as in [5]) the Jordan form. The main tool is Hopf's inequality. As it has been used for bounding the eigenvalues $\neq \rho$, it is quite natural to use it for convergence proofs, too.

2. DEFINITIONS; TWO LEMMAS

An $N \times N$ matrix $B = (b_{ik})$ is called positive (non-negative) if $b_{ik} > 0$ ($\geq 0$), $i, k = 1, \ldots, N$. We write $B > 0$ ($\geq 0$). For vectors, $y > 0$, $y \geq 0$ are defined in an analogous way.

LINEAR ALGEBRA AND ITS APPLICATIONS 15, 235-242 (1976) 235

For a pair of vectors $x, y$ with $y > 0$, we define

$$\max\left(\frac{x}{y}\right) = \max_i \frac{x_i}{y_i}, \quad \min\left(\frac{x}{y}\right) = \min_i \frac{x_i}{y_i},$$

$$\text{osc}\left(\frac{x}{y}\right) = \max\left(\frac{x}{y}\right) - \min\left(\frac{x}{y}\right).$$

Hopf's inequality [1, 3, 6] states: For $B > 0$ and any pair of vectors $x, y$, where $y > 0$,

$$\text{osc}\left(\frac{Bx}{By}\right) \leq N(B)\text{osc}\left(\frac{x}{y}\right). \quad (1)$$

Here

$$N(B) = \frac{\sqrt{K(B)} - 1}{\sqrt{K(B)} + 1}$$

and

$$K(B) = \sup_{u > 0, \ v > 0} \left\{ \max\left(\frac{Bu}{Bv}\right) \max\left(\frac{Bv}{Bu}\right) \right\}. \quad (2)$$

It is obvious that

$$N(tB) = N(B), \quad t > 0, \quad (3)$$

$$N(pq^T) = 0, \quad p > 0, \quad q > 0, \quad (4)$$

$$N(D_1BD_2) = N(B), \quad (5)$$

where $D_i$ ($i = 1, 2$) are diagonal matrices with positive diagonal entries. A bound for $N(B)$ is [3, 6]

$$N(B) \leq \frac{m_1 - m_2}{m_1 + m_2}, \quad m_1 = \max_{i,k} b_{ik}, \quad m_2 = \min_{i,k} b_{ik}. \quad (6)$$
SPECTRAL RADIUS OF A NON-NEGATIVE MATRIX

Lemma 1. Let \( p > 0, \ q > 0, \ \bar{p} = \min p_i, \ \bar{q} = \min q_i, \) and \( B = (b_{ik}) \) be a positive matrix such that

\[
|b_{ik} - p_i q_k| \leq \varepsilon. \tag{7}
\]

Then

\[
N(B) \leq \frac{\varepsilon}{\bar{p} \bar{q}}. \tag{8}
\]

Proof. Define \( \tilde{B} = (\tilde{b}_{ik}), \ \tilde{b}_{ik} = b_{ik} / p_i q_k, \ \tilde{\varepsilon} = \varepsilon / \bar{p} \bar{q}. \) Then by (7)

\[
\max \tilde{b}_{ik} < 1 + \tilde{\varepsilon}, \quad \min \tilde{b}_{ik} > 1 - \tilde{\varepsilon},
\]

and hence by (5) and (6)

\[
N(B) = N(\tilde{B}) \leq \tilde{\varepsilon}. \tag{9}
\]

Remark. By taking \( p_i = (m_1 + m_2) / 2, \ q_i = 1, \ \varepsilon = (m_1 - m_2) / 2 \) in Lemma 1, (8) yields the bound (6).

Lemma 2. For a given number \( \lambda_0 > \rho \) there is an \( M > 0 \) such that

\[
N((\lambda I - A)^{-1}) \leq M(\lambda - \rho), \quad \rho < \lambda < \lambda_0. \tag{9}
\]

Proof. The adjoint \( \text{adj}(B) \) of a square matrix \( B \) satisfies the relation [4, p. 13]

\[
B \text{adj}(B) = \text{adj}(B)B = (\det B)I
\]

In particular, for \( \lambda > \rho, \)

\[
(\lambda I - A)^{-1} = \frac{1}{\det(\lambda I - A)} \text{adj}(\lambda I - A).
\]

Hence by (3),

\[
N((\lambda I - A)^{-1}) = N(\text{adj}(\lambda I - A)). \tag{10}
\]
On the other hand,

$$\text{adj}(\rho I - A) = pq^T,$$

where $q > 0$, $A^Tq = \rho q$, $q$ suitably normalized. Equation (9) follows now from Lemma 1. \[\Box\]

3. THE ITERATIVE PROCEDURE

We define

$$\| x \| = \max \left( \frac{x}{p} \right).$$

Let $\{B_n\}$, $n = 0, 1, \ldots$ be a sequence of positive matrices commuting with $A$. Assume the existence of $\gamma$ such that

$$N(B_n) \leq \gamma < 1, \quad n = 0, 1, \ldots.$$  \tag{12}

For given $x_0 > 0$, define iteratively

$$\tilde{x}_{n+1} = B_n x_n,$$  \tag{13}

$$x_{n+1} = \frac{\tilde{x}_{n+1}}{\| \tilde{x}_{n+1} \|},$$  \tag{14}

$$\bar{\lambda}_{n+1} = \max \left( \frac{Ax_{n+1}}{x_{n+1}} \right), \quad \underline{\lambda}_{n+1} = \min \left( \frac{Ax_{n+1}}{x_{n+1}} \right).$$  \tag{15}

$\bar{\lambda}_0, \underline{\lambda}_0$ are defined analogously.

We prove first some useful relations:

LEMMA 3. For $n = 1, 2, \ldots,$

$$\bar{\lambda}_n - \rho \leq \rho \frac{\text{osc}(x_n/p)}{1 - \text{osc}(x_n/p)} \leq C\rho (\bar{\lambda}_n - \rho),$$  \tag{16}

$$\rho - \underline{\lambda}_n \leq \rho \text{osc}\left( \frac{x_n}{p} \right) \leq \tilde{C} (\rho - \underline{\lambda}_n),$$  \tag{17}

where $C, \tilde{C}$ depend on $A$, and $C$ also on $\bar{\lambda}_0$. 
SPECTRAL RADIUS OF A NON-NEGATIVE MATRIX

Proof. From $\|x_n\| = 1$, $n > 0$ we get

$$1 - \text{osc} \left( \frac{x_n}{p} \right) \leq \frac{x_{n,i}}{p_i} \leq 1.$$ 

Hence for suitable $s$

$$\lambda_n - \rho = \sum_k a_{ik} \frac{p_k}{p_s} \left( \frac{x_{n,k}}{x_{n,s}} - 1 \right) \leq \rho \left( \frac{1}{1 - \text{osc}(x_n/p)} - 1 \right).$$

showing the left inequality of (16). The left inequality of (17) follows in an analogous way. For the other inequalities we use a result in [2, Folgerung 2, p. 72]. Let $x > 0$, $z > 0$, and $Ax \leq \alpha x$, $A z \geq \beta z$, and choose $i$ so that $x_i/z_i$ is minimal. For any $k \neq i$ there is an $s < n - 1$ such that $a_{ik} = (A^s)_{ik} > 0$ and

$$\frac{x_i}{z_i} \leq \frac{1}{\alpha - \beta \frac{z_i}{z_k}} \frac{x_i}{z_i} \leq \left( 1 + \frac{\alpha - \beta}{a_{ik}} \frac{z_i}{z_k} \right) \frac{x_i}{z_i} \leq \left( 1 + \frac{\alpha - \beta}{a_{ik}} \frac{x_k}{x_i} \right) \frac{x_i}{z_i} \leq \frac{x_k}{x_i} \leq \frac{x_k}{z_i} \left( 1 - \frac{\alpha - \beta}{a_{ik}} \frac{x_i}{x_k} \right) \frac{z_i}{x_i} \leq \frac{z_i}{x_i}. \quad (18)$$

and

$$\left( 1 - \frac{\alpha - \beta}{a_{ik}} \frac{x_i}{x_k} \right) \frac{z_i}{x_i} \leq \frac{z_i}{x_i} \leq \frac{z_i}{x_i}. \quad (19)$$

Taking $x = x_n$, $\alpha = \lambda_n$, $z = p$, $\beta = \rho$ in (18), we get

$$\min \left( \frac{x_n}{p} \right) \leq \frac{x_{n,k}}{p_k} \leq \left[ 1 + C(\lambda_n - \rho) \right] \min \left( \frac{x_n}{p} \right)$$

for suitable $C$ depending on an upper bound for $\lambda_n$, $n = 1, 2, \ldots$. According to Theorem 1 such a bound is provided by $\lambda_0$. Thus

$$\text{osc} \left( \frac{x_n}{p} \right) \leq C(\lambda_n - \rho) \left[ 1 - \text{osc} \left( \frac{x_n}{p} \right) \right],$$

yielding the right inequality in (16).

Taking $x = p$, $\alpha = \rho$, $z = x_n$, $\beta = \lambda_n$ in (19), we get for a suitable $\tilde{C}$

$$\left[ 1 - \tilde{C} (\rho - \lambda_n) \right] \max \left( \frac{x_n}{p} \right) \leq \frac{x_{n,k}}{p_k} \leq \max \left( \frac{x_n}{p} \right).$$
or

\[ \text{osc} \left( \frac{x_n}{p} \right) \leq C (\rho - \underline{\lambda}_n). \]

This is the second inequality in (17).

\[ \text{Theorem 1. Consider the procedure (13)-(15). For } n = 0, 1, 2, \ldots, \]

\[ \underline{\lambda}_n \leq \underline{\lambda}_{n+1} \leq \rho \leq \overline{\lambda}_{n+1} \leq \overline{\lambda}_n, \quad (20) \]

\[ \lim \underline{\lambda}_n = \lim \overline{\lambda}_n = \rho, \quad (21) \]

\[ \lim x_n = p. \quad (22) \]

If \( x_n \neq p \) for all \( n \), then the inequalities of (20) are strict.

\[ \text{Proof. If we multiply the relation} \]

\[ \underline{\lambda}_n x_n \leq Ax_n \leq \overline{\lambda}_n x_n \]

by \( B_n \) and use \( B_n A = AB_n \), we get

\[ \underline{\lambda}_n \tilde{x}_{n+1} \leq A \tilde{x}_{n+1} \leq \overline{\lambda}_n \tilde{x}_{n+1} \]

and hence \( \underline{\lambda}_n \leq \underline{\lambda}_{n+1}, \overline{\lambda}_{n+1} \leq \overline{\lambda}_n \). If \( x_n \neq p \), then \( \overline{\lambda}_n x_n - Ax_n \neq 0 \); hence \( \overline{\lambda}_n \tilde{x}_{n+1} - A \tilde{x}_{n+1} \geq 0 \) and \( \underline{\lambda}_{n+1} < \overline{\lambda}_n \). Similarly \( \underline{\lambda}_n < \underline{\lambda}_{n+1} \). The remaining inequalities \( \underline{\lambda}_n \leq \rho \leq \overline{\lambda}_n \) follow from the quotient theorem (e.g., [4], II, 5.5.2). From (16), (17) we infer the strict inequalities for \( x_n \neq p \). Now

\[ \overline{\lambda}_{n+1} - \underline{\lambda}_{n+1} = \text{osc} \left( \frac{Ax_{n+1}}{x_{n+1}} \right) = \text{osc} \left( \frac{B_n Ax_n}{B_n x_n} \right) \leq N(B_n) \text{osc} \left( \frac{Ax_n}{x_n} \right) \]

\[ = N(B_n)(\overline{\lambda}_n - \underline{\lambda}_n). \quad (23) \]
From (12) and (20), we infer (21). From (17), we get
\[
\lim_{n \to \infty} \text{osc} \left( \frac{x_n}{p} \right) = 0
\]
or
\[
\lim_{n \to \infty} \frac{x_{n,i}}{p_i} = 1, \quad i = 1, \ldots, N.
\]
Hence, we get (22).

In the case of \( A > 0 \), taking \( B_n = A \), Theorem 1 gives the convergence of the usual power method. If only \( A^m > 0 \) for a suitable integer \( m \), i.e., if \( A \) is primitive, the proof given above can be easily adapted to yield the same result. In fact,
\[
\lambda_{n+m} - \lambda_{n+m} = \text{osc} \left( \frac{A^m A x_n}{A^m x_n} \right) \leq N (A^m) \left( \lambda_n - \lambda_n \right).
\]
More interesting is the case
\[
B_n = (\bar{\lambda}_n I - A)^{-1}, \quad n = 0, 1, \ldots
\]
(24)
If we start with an \( x_0 \) such that \( Ax_0 \neq \rho x_0 \), then \( x_n \neq p, \bar{\lambda}_n > \rho, B_n > 0 \) for all \( n \), as can be proved by induction.

Hence, Theorem 1 can be applied and gives the convergence of the inverse iteration procedure considered by Noda [5]. Additionally, we have the following statement about the rate of convergence:

**Theorem 2.** In the iteration procedure (13)–(15) with
\[
B_n = (\bar{\lambda}_n I - A)^{-1},
\]
the sequences \( \{\lambda_n\} \), \( \{\bar{\lambda}_n\} \) converge quadratically to \( \rho \) and the \( \{x_n\} \) quadratically to the eigenvector \( p \).

**Proof.** From (23) and (9) we get
\[
\bar{\lambda}_{n+1} - \lambda_{n+1} \leq M (\bar{\lambda}_n - \rho) (\bar{\lambda}_n - \lambda_n) \leq M (\bar{\lambda}_n - \lambda_n)^2,
\]
i.e., \( \{ \bar{\lambda}_n - \lambda_n \} \) converges quadratically to zero. It is now obvious from (16) and (17) that the sequences

\[
\{ \bar{\lambda}_n - \rho \}, \quad \{ \rho - \lambda_n \}, \quad \text{osc} \left( \frac{x_n}{\rho} \right)
\]

also converge quadratically.

\[ \square \]

Note added in proof.

The author learned that Theorem 2 has also been proved in Stephen M. Robinson-Karl Nickel: Computation of the Perron root and vector of a nonnegative matrix, MRC Technical Summary Report \#1100, September 1970, Mathematics Research Center, University of Wisconsin-Madison, Madison, Wisconsin 53706.

REFERENCES


Received 22 August 1975; revised 17 November 1975