A Note on Characterizations of Irreducibility of Nonnegative Matrices

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ABSTRACT

Three sufficient conditions for the irreducibility of a matrix A are given, which for nonnegative A are also necessary.

In this note we shall prove the

THEOREM. *Let* $A \geq 0$ *be a nonnegative* $n \times n$ *matrix,*

$$x = (x_1, \dots, x_n)^T > 0, \qquad q_i = (Ax)_i / x_i, \qquad i = 1, \dots, n,$$

$$D = \operatorname{diag}(q_1, \dots, q_n).$$

Then the following conditions are equivalent:

(a) A *is irreducible.*
(b) $\operatorname{Rank}(A-D) = n-1$. *There exists* $z > 0$ *such that* $A^T z = Dz$.
(c) $\operatorname{Rank}(A-D) = n-1$. *There exist* z, $z_i \neq 0$, $i = 1, \dots, n$, *such that* $A^T z = Dz$.
(d) $A^T z = Dz$, $z \neq 0 \Rightarrow z_i \neq 0$, $i = 1, \dots, n$.

Proof.

(a)→(b): For s sufficiently large, $B = A - D + sI$ is nonnegative irreducible and $Bx = sx$. The Perron-Frobenius theorem states among other results that s is a simple eigenvalue of B, so that $\operatorname{Rank}(A-D) = n-1$, and that there is a vector $z > 0$ with $z^T B = s z^T$, or $z^T(D-A) = 0$.

(b)→(c): Trivial.

(c)→(d): From $\operatorname{Rank}(A^T - D) = n-1$ it follows that z is unique up to a factor. This shows (d).

(d)→(a): If A is reducible, we may assume $A = \begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix}$ with an $s\times s$ matrix A_1 $(s<n)$. Now $(A_1 - D_1)\tilde{x} = 0$, where $D_1 = \operatorname{diag}(q_1,\dots,q_s)$, $\tilde{x}^T = (x_1,\dots,x_s)$. Hence there is a $\tilde{z}\neq 0$, $\tilde{z}^T(A_1 - D_1) = 0$. It follows that

$$A^T\begin{pmatrix}\tilde{z}\\0\end{pmatrix} = D\begin{pmatrix}\tilde{z}\\0\end{pmatrix}.$$

This contradicts (d). ■

REMARK 1. The equivalence of (a), (b) and (c) in the special case $x = (1,\dots,1)^T$ was shown in [1]. The general case can be reduced to the special case by considering $X^{-1}AX$, where $X = \operatorname{diag}(x_1,\dots,x_n)$, but we feel that the proof given here is shorter and more elementary.

REMARK 2. The nonnegativity of A was used only in (a)→(b). Hence each of the conditions (b), (c), (d) imply A irreducible also for general A. Observe that it suffices here to require $x_i \neq 0$, $i = 1,\dots,n$.

REFERENCES

1 I. M. Chakravarti, On a characterization of irreducibility of a nonnegative matrix, *Linear Algebra Appl.* **10** (1975), 103–109.

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