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Legendre Padé Approximants in $\pi\mathcal{N}$ Scattering (*).

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Summary. — Padé approximants are expected to probe the singularity structure of scattering amplitudes. We apply Padé approximants for Legendre series to $\pi\mathcal{N}$ amplitudes, which are obtained from phase shifts. The imaginary parts were calculated from the phase-shift analysis of Almehed and Lovelace, the real parts from fixed-t dispersion relations directly and from the phase-shift analysis of Nielsen and Oades. For all amplitudes (except for the isospin-odd Nielsen-Oades) the poles of the Padé approximants are outside the region of analyticity in the Mandelstam plane. The Padé approximants and the truncated Legendre series for the imaginary parts deviate from each other only up to 1% even close to the boundary of the double-spectral function (except for $B^{(+)}$).

1. – Introduction.

Padé approximants are best known through their powerful applicability in the summation of Taylor series. In particular some progress has been made in the summation of the power series of strong-interaction field theories (1). While in the latter case one is summing the Taylor series in a domain of analyticity of the function, Padé approximants are used in statistical mechanics to reconstruct the singularities of the function (2), for which purpose higher-order approximants are used, however.

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(1) J. L. BASDEVANT: in *Padé Approximants*, edited by P. R. GRAVES-MORRIS (London and Bristol, 1973), p. 77.

(2) M. E. FISHER: in *Padé Approximants and their Applications*, edited by P. R. GRAVES-MORRIS (London and New York, N. Y., 1973), p. 159.

Many problems of scattering theory involve Legendre series or related series. Padé approximants for Legendre series, however, have only been introduced recently ([3,4]). Their excellent convergence properties, as well as their ability of reproducing the singularity structure of a given function have been demonstrated by numerical examples in ref. ([4]). For these examples the Legendre Padé approximants could be used to analytically continue the functions into their regions of analyticity beyond the regions of convergence of the original Legendre series. Moreover all poles were lying on the cuts of the functions under consideration, thus simulating the cuts. Even the residues of the poles could give some information about the imaginary part on the cut.

Though there exist no general proofs of these properties for certain classes of functions, one may hope that the Legendre Padé approximants for the amplitudes of physical processes behave in the same way.

If this is true, it is possible to study the analytic properties which are implied by a phase-shift analysis by looking at the pole distribution of the Legendre Padé approximants to the Legendre series. This analyticity test has advantages compared to the canonical test via dispersion relations:

i) dispersion relations test the dispersive and absorptive parts of the amplitudes at the same time and do not allow separate tests for each part of the amplitude,

ii) as input to dispersion relations usually extra information such as coupling constants or knowledge on high-energy behaviour is needed,

iii) from the position of the poles on the cuts important regions of the cuts are indicated.

The most detailed phase-shift analyses exist for pion-nucleon scattering. The aim of this paper is to study the analytic properties implied by these phase shifts via Legendre Padé approximants. Furthermore we want to compare the truncated Legendre series obtained from phase shifts and their Legendre Padé approximants outside the region of convergence of the original Legendre series. It is, however, not our intention to use the data to directly analyse $\pi\mathcal{N}$ scattering in terms of Padé approximants, which could also be done.

The plan of the paper is as follows. In Sect. **2** we define the Legendre Padé approximants and explain their connection to the usual Padé approximants. In Sect. **3** we give the Legendre expansions of the $\pi\mathcal{N}$ amplitudes and their analyticity domains as expected from the Mandelstam representation. Section **4** contains the pole and zero distributions of the Padé approximants to the Legendre series, where the absorptive parts were calculated from the Lovelace-

([3]) J. T. HOLDEMAN jr.: *Math. Comput.*, **23**, 275 (1969); J. FLEISCHER: *Nucl. Phys.*, **37** B, 59 (1972) (Erratum: *Nucl. Phys.*, **44** B, 641 (1972)).
([4]) J. FLEISCHER: *Journ. Math. Phys.*, **14**, 246 (1973).

Almehed phase shifts [5], the dispersive parts from a set of Nielsen-Oades phase shifts [6]. For comparison, the dispersive parts as obtained from fixed-t dispersion relations are also analysed.

2. – Padé approximants for Legendre series.

As Padé approximants for series in orthogonal polynomials are not yet standard, it will be useful to explain briefly the main ideas. For completeness we first present the definition of Padé approximants for Taylor series: given a function $f(z)$ in terms of its (not necessarily convergent) Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \,, \tag{2.1}$$

the $[N/M]$ Padé approximant is defined as

$$[N/M](z) = P_N(z)/Q_M(z) \,, \tag{2.2}$$

where the polynomials $P_N(z)$ and $Q_M(z)$ are obtained from

$$f(z)Q_M(z) - P_N(z) = Az^{M+N+1} + \dots \,, \tag{2.3}$$

or equivalently

$$f(z) - P_N(z)/Q_M(z) = A'z^{M+N+1} + \dots \,. \tag{2.4}$$

These approximants are well established as a powerful means for improving the convergence of Taylor series and analytic continuation of the corresponding functions. It seems important to extend this method on functions given in terms of orthogonal polynomials [4,7]. In particular Legendre series are of special interest in physics. Given a series in Legendre polynomials

$$f(z) = \sum_{l=0}^{\infty} a_l P_l(z) \,, \tag{2.5}$$

one cannot trivially re-order the series in a Taylor series and for its analytic continuation simply apply Padé approximants for Taylor series. The reason

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([5]) S. Almehed and C. Lovelace: *Nucl. Phys.*, **40** B, 157 (1972).
([6]) H. Nielsen and G. C. Oades: *Nucl. Phys.*, **49** B, 573 (1972).
([7]) J. Fleischer: in *Padé Approximants*, edited by P. R. Graves-Morris (London and Bristol, 1973), p. 126; see also J. Fleischer: in *Padé Approximants and their Applications*, edited by P. R. Graves-Morris (London and New York, N. Y., 1973), p. 69.

is that each coefficient of the obtained Taylor series could only be expressed as a series over all (even or odd, respectively) Legendre coefficients. Therefore, taking into account one more coefficient in the Legendre series would change the lower coefficients in the Taylor series and hence render the corresponding Padé approximants for Taylor series irrelevant.

One can, however, set up Padé approximants for Legendre series in complete analogy to eqs. (2.3) and (2.4). The complication in this case stems from the fact that the product of two Legendre polynomials can only be expressed as the following sum over Legendre polynomials:

$$P_i(z)P_k(z)=\sum_{l=|i-k|}^{i+k} a_l^{(i,k)}P_l(z)\,. \tag{2.6}$$

As a consequence of this, the definitions of Padé approximants in analogy to eqs. (2.3) and (2.4), respectively, do not yield identical approximants. While the « cross-multiplied » approximants (7)

$$\begin{aligned}(a_0P_0+a_1P_1+\dots+a_LP_L)(d_0P_0+d_1P_1+\dots+d_MP_M)&=\\ =n_0P_0+n_1P_1+\dots+n_NP_N+0+\dots+0+\bar{n}_{M+N+1}P_{M+N+1}+\dots\end{aligned} \tag{2.7}$$

$(L=2M+N)$, defined in analogy to eq. (2.3), require for the determination of the coefficients n_i, d_k $(i=0,\dots,N;\,k=0,\dots,M)$ only the solution of a linear system of equations, they do not have the property that the first expansion coefficients of the fraction

$$\frac{R_N(z)}{S_M(z)}=\frac{n_0P_0(z)+n_1P_1(z)+\dots+n_NP_N(z)}{d_0P_0(z)+d_1P_1(z)+\dots+d_MP_M(z)} \tag{2.8}$$

are the same as in the originally given function. If one wants to impose this condition in analogy to (2.4)

$$f(z)-\frac{R_N(z)}{S_M(z)}=\bar{a}_{M+N+1}P_{M+N+1}(z)+\dots, \tag{2.9}$$

one has explicitly to project the denominator to obtain its Legendre series, which clearly yields logarithmic terms and hence results in a transcendental system of equations for the coefficients d_i $(i=1,\dots,M)$.

It has been shown (4) that these equations can be solved and, as found from numerical experiments, the solutions seem to be unique in general. The latter (« properly expanded » (7)) approximants also have better convergence properties than the cross-multiplied ones. In ref. (4) their complete analogy to Padé approximants for Taylor series has been shown. A simple proof of convergence follows the same lines as for Taylor series: once the Padé approximants are bounded with rising order, they converge. Further theorems of convergence, as proven *e.g.* for Stieltjes functions, will be more difficult to obtain in our case because of the nonlinearlity of the equations. The numerical experience seems to justify, however, their applicability on the same grounds as for ordinary Padé approximants for Taylor series.

3. – The $\pi\mathcal{N}$ amplitudes.

3'1. *The Legendre expansions.* – As a consequence of the nucleon spin the $\pi\mathcal{N}$ partial-wave expansions are usually given in terms of the first derivatives of Legendre polynomials and not in Legendre polynomials directly. Rewriting these partial-wave expansions one obtains

$$\begin{aligned}(3.1)\qquad \frac{B}{4\pi} = \frac{1}{q^2}\sum_{l=0}^{\infty}\Big\{&f_{l+}(s)\sum_{k=0}^{l}(2k+1)P_k(x)[-M+(-1)^{k+l}E]-\\ &-f_{l-}(s)\sum_{k=0}^{l-1}(2k+1)P_k(x)[-M+(-1)^{k+l}E]\Big\},\end{aligned}$$

$$\begin{aligned}(3.2)\qquad \frac{A'}{4\pi}(4M^2-t) = 2W\sum_{l=0}^{\infty}\Big\{&f_{l+}(s)(l+1)[(E+M)P_l(x)-(E-M)P_{l+1}(x)]+\\ &+f_{l-}(s)l[(E+M)P_l(x)-(E-M)P_{l-1}(x)]\Big\},\end{aligned}$$

where $f_{l\pm}(s)$ are the s-channel partial-wave amplitudes, $x=\cos\theta_s$, M, E and q are the nucleon mass, c.m. energy and momentum, $W=\sqrt{s}$ (*). For convenience we have multiplied A' by $4M^2-t$. The analytic properties of the amplitude remain unchanged by this factor.

The coefficients a'_l of the new amplitude

$$(3.3)\qquad \frac{A'}{4\pi}(4M^2-t) = \sum_{l=0}^{\infty} a'_l P_l(x)$$

simply read

$$\begin{aligned}(3.4)\qquad a'_l = 2W\{&(l+1)(E+M)f_{l+}-l(E-M)f_{l-1,+}+\\ &+l(E+M)f_{l-}-(l+1)(E-M)f_{l+1,-}\}.\end{aligned}$$

The Legendre coefficients of the B-amplitude

$$(3.5)\qquad \frac{B}{4\pi} = \sum_{l=0}^{\infty} b_l P_l(x),$$

(*) We use $m_\pi=\hbar=c=1$ throughout the paper.

however, cannot be expressed by a finite number of partial-wave amplitudes:

$$b_l = \frac{2l+1}{q^2}\left\{f_{l+}(E-M) + \sum_{k=l+1}^{\infty}[-M+(-1)^{k+l}E](f_{k+}-f_{k-})\right\}. \tag{3.6}$$

This means no complication, since the usual phase-shift analyses determine only a finite number of partial waves. The higher ones are put equal to zero at the outset.

3'2. *The analyticity domains.* – From the Mandelstam representation we expect the absorptive parts of the amplitudes to be analytic functions in t for fixed $s>(M+1)^2$ and $\tau_-^1(s)<t<\tau_+^1(s)$ for $(M+1)^2<s<(M+2)^2$; $\max(\tau_-^1(s), \tau_-^2(s))<t<\min(\tau_+^1(s), \tau_+^2(s))$ for $s\geqslant(M+2)^2$, where the τ's are the boundaries of the double-spectral function domains (8)

$$\tau_-^1(s) = M^2+6-s+2\,\frac{M^2+2-s}{q^2} - 4\sqrt{\left(1+\frac{1}{q^2}\right)(M^2+2-s)\left(1+\frac{M^2+2-s}{4q^2}\right)}\,, \tag{3.7}$$

$$\tau_-^2(s) = \tau_-^1(2M^2+2-s-\tau_-^2(s))\,, \tag{3.8}$$

$$\tau_+^1(s) = 16+\frac{16}{q^2}\,, \tag{3.9}$$

$$\tau_+^2(s) = 4+16\,\frac{s+3(M^2-1)}{[s-(M+2)^2][s-(M-2)^2]} \tag{3.10}$$

and

$$q^2(s) = \frac{1}{4s}[s-(M+1)^2][s-(M-1)^2]\,. \tag{3.11}$$

The dispersive parts of the amplitudes are—according to the Mandelstam representation—analytic functions in t for fixed $s>(M+1)^2$ and

$$(M-1)^2-s<t<4\,, \tag{3.12}$$

where the lower bound is due to the start of the u-cut $(u=(M+1)^2)$, the upper bound is due to the start of the t-cut $(t=4)$. Apart from the cuts, the dispersive parts have a pole at

$$u=M^2\,,\qquad i.e.\ t=M^2+2-s\,.$$

(8) W. R. Frazer and J. R. Fulco: *Phys. Rev.*, **117**, 1603 (1960).

Inside their analyticity domains the absorptive and dispersive parts are equal to the imaginary and real parts of the amplitudes, respectively.

The partial-wave expansions of the dispersive and absorptive parts converge inside the smallest ellipses in the $\cos\theta_s$-plane, which have foci at ± 1 and touch the boundaries of the analyticity domains, the transformation from the t-plane to the $\cos\theta_s$-plane being

$$\cos\theta_s = 1 + t/2q^2 . \tag{3.13}$$

It is clear that the usual partial-wave expansions cannot make full use of the analytic properties of the amplitudes. The expansions for the dispersive parts *e.g.* are already limited by $|\cos\theta_s| = 1 + 4/2q^2$.

4. – Numerical results.

Studying the pole and zero distributions of Padé approximants for $\pi\mathcal{N}$ scattering amplitudes with experimental input for the phase shifts, we observe great similarity with what has been found in the study of Padé approximants for Taylor series in the presence of noise [9]. We want to stress the important nontrivial numerical result, that the nonlinear equations determining the Legendre Padé approximants have only one solution with poles outside the physical region (see also ref. [4]).

Our Padé approximants have in general no poles in the region of analyticity except for pole-zero pairs—the so-called « doublets » [9]—which are very unstable with the order of the approximant. There occur, however, stable poles with no accompanying zeros, indicating an important region of the cut.

Concerning the doublets, it is interesting to note that the distance between the pole and zero in a doublet becomes smaller, the smaller the doublet's distance from the physical region is. So it may happen that a doublet is migrating (as a function of s) from one region of the double-spectral function ($t<0$) through the analyticity domain to the other ($t>0$). It looks as if the zero is catching the pole in a region where the pole should not be and lets it free where the amplitude is not analytic any more (see, *e.g.*, Fig. 3, $105 \leqslant s \leqslant 110$).

We have used the Lovelace-Almehed phase shifts [5] except for very low energies ($s<64$). In this low-energy region the phase shifts are not well known. Furthermore, to calculate the amplitudes for a negative-t value outside the physical region, one has to analytically continue to much higher $\cos\theta_s$ values.

4.1. *The imaginary parts.* In Fig. 1-4 we present our results for the pole-zero distributions of the Padé approximants for the imaginary parts of the

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[9] J. L. BASDEVANT: in *Methods in Subnuclear Physics*, Vol. 4, part 2 (Herceg-Novi, 1968), p. 129.

TABLE I. – $s[m_\pi^2]$ *ranges for the Padé approximants shown in Fig.* 1-4.

	[1/1]	[2/1]	[1/2]	[2/2]	[3/2]	[3/3]
Im $A'^{(+)}$	$60.5 \div 76.4$			$77.0 \div 91.9$	$92.5 \div 117.3$	$117.8 \div 220.3$
Im $A'^{(-)}$	$60.5 \div 76.4$			$77.0 \div 91.9$	$92.5 \div 117.3$	$117.8 \div 161.1$
Im $B^{(+)}$	$67.4 \div 75.8$		$76.4 \div 91.9$	$92.5 \div 117.3$	$117.8 \div 218.6$	
Im $B^{(-)}$	$75.2 \div 75.8$	$76.4 \div 91.9$		$92.5 \div 117.3$	$117.8 \div 185.9$	

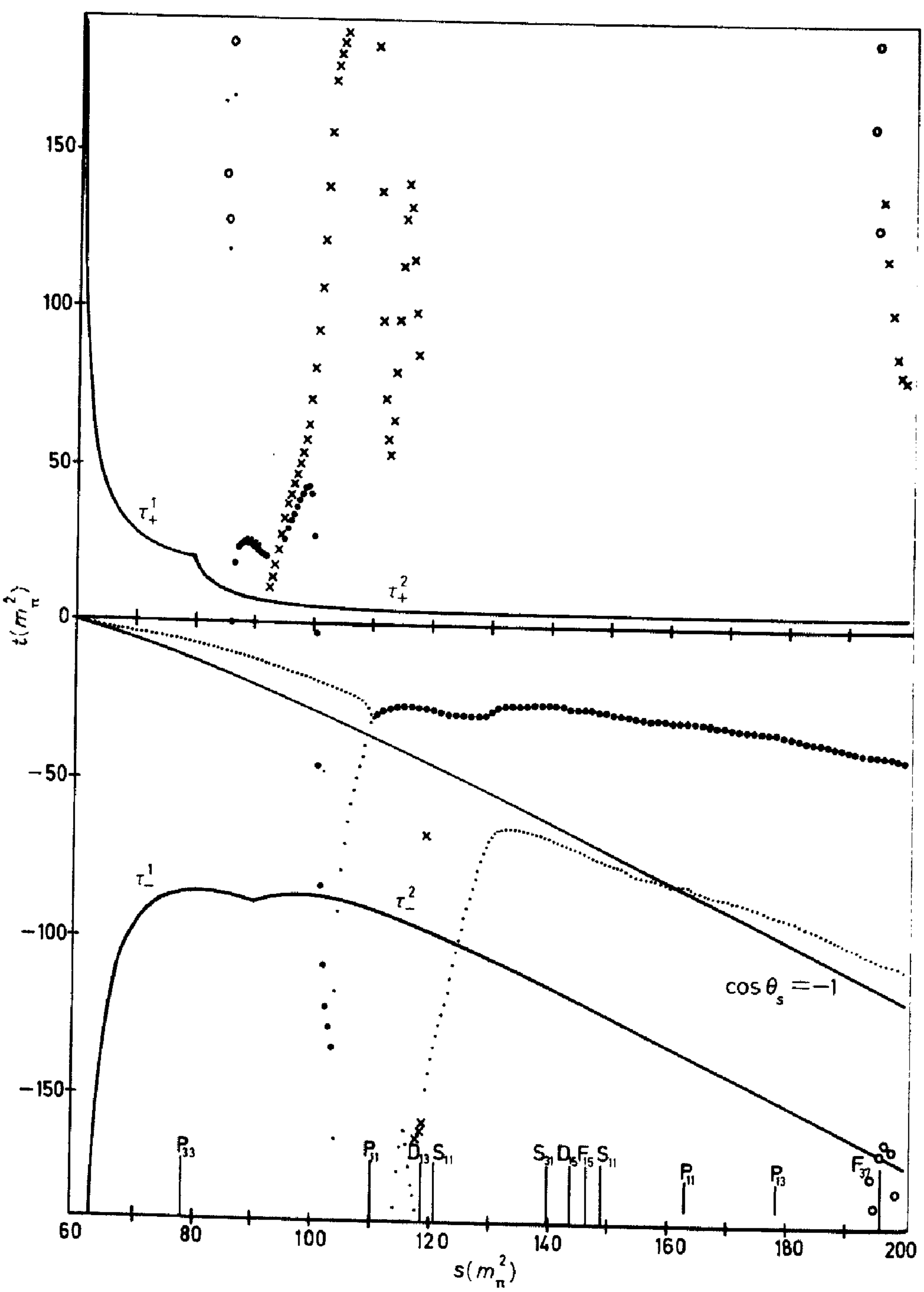

Fig. 1. – Pole-zero distribution in the Mandelstam plane for Im $A'^{(+)}$. For all energies approximants of highest possible order have been calculated. o, ·, × and • indicate real pole, real zero, real part of complex pole and real part of complex zero, respectively.

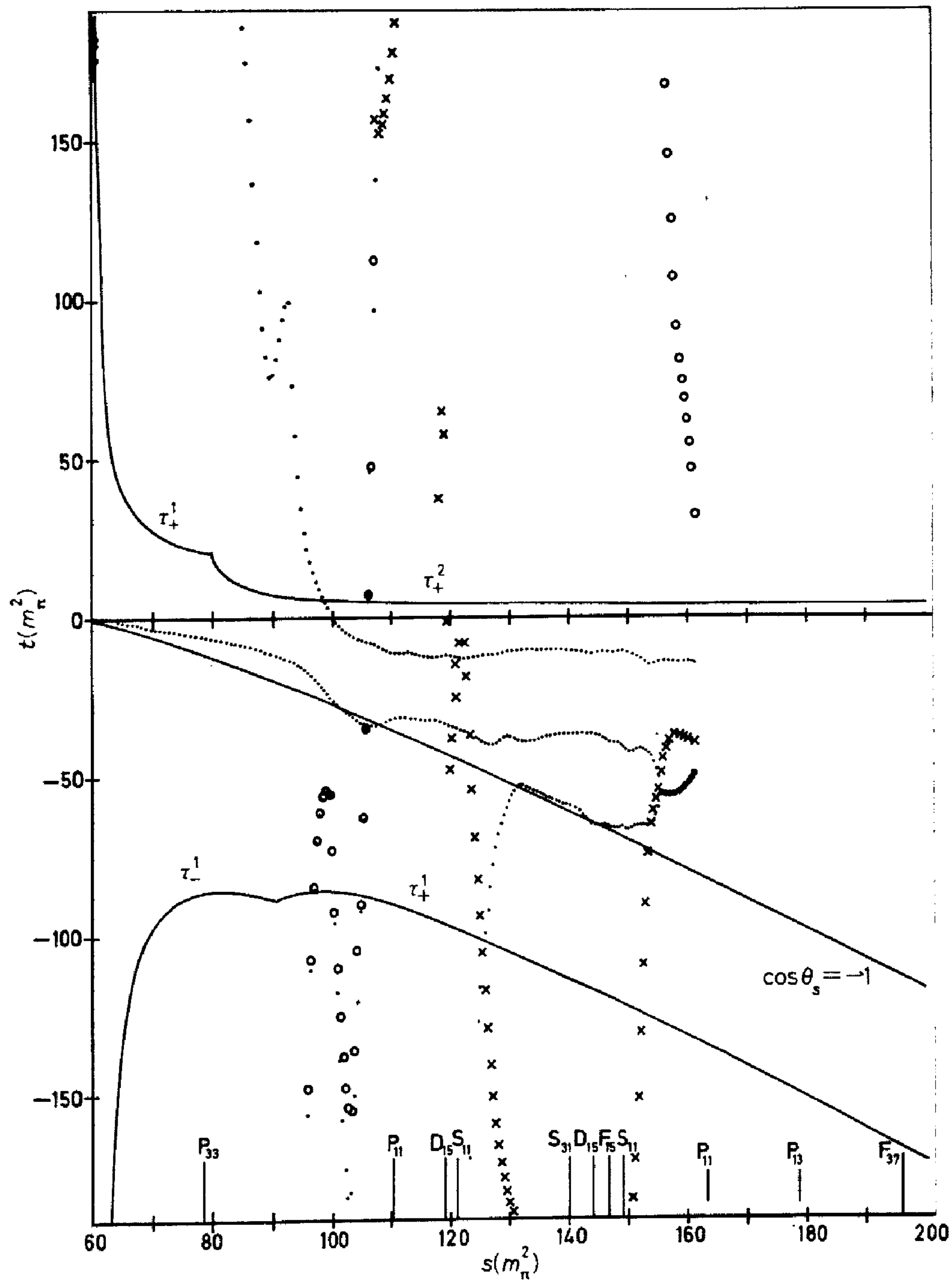

Fig. 2. – Pole-zero distribution for Im $A'^{(-)}$, same notation as in Fig. 1.

various amplitudes ($A'^{(\pm)}, B^{(\pm)}$ (*)). Part of the poles and zeros may of course be out of the range of the Figures. In general we calculated the highest possible order. As the number of available partial waves increases with energy, the highest possible order does so too. Table I specifies the orders of Padé approximants, calculated for various energies. At the highest energies ($\leqslant s = 250$)

(*) $^{(\pm)}$ means isospin even or odd.

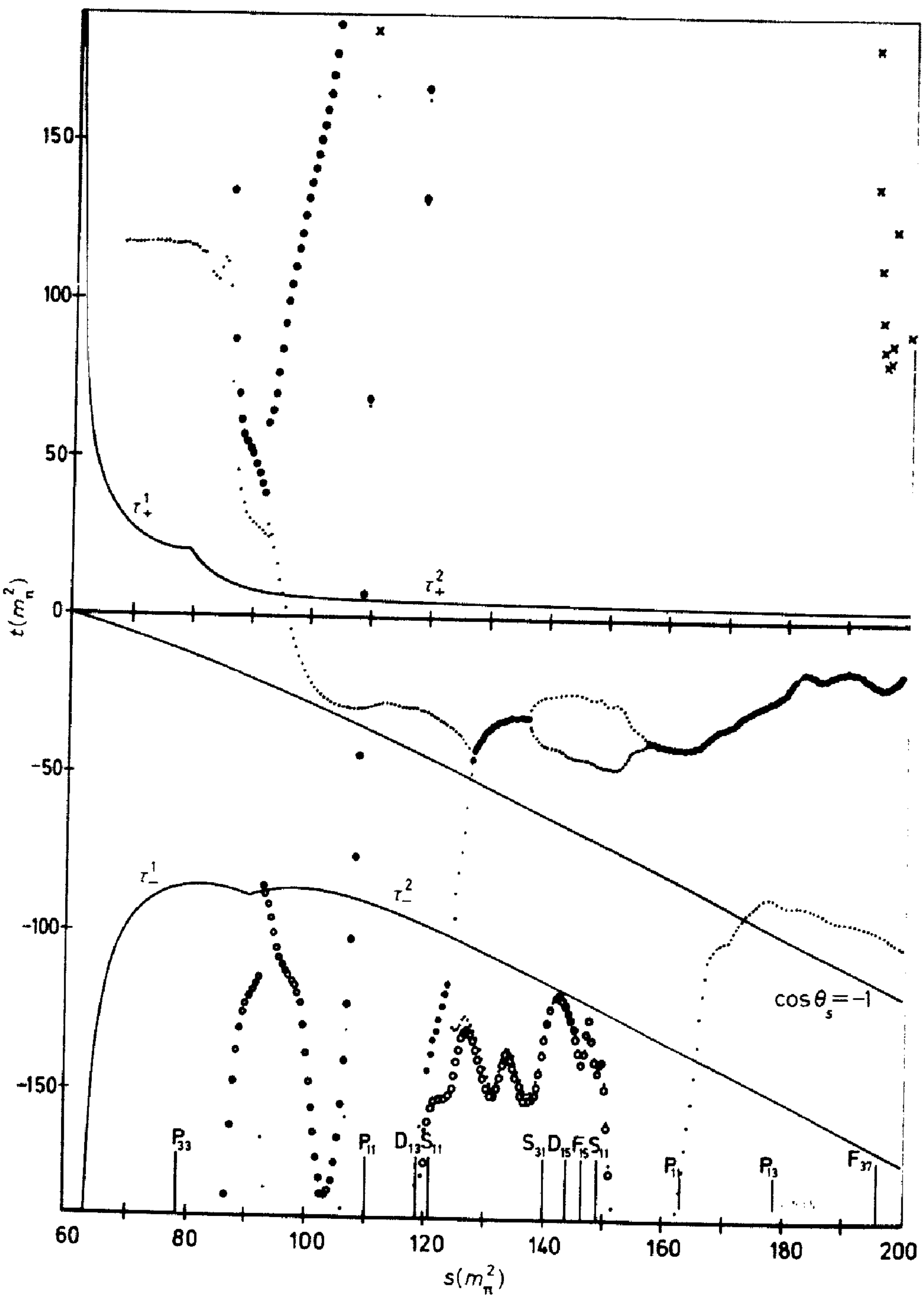

Fig. 3. – Pole-zero distribution for $\operatorname{Im} B^{(+)}$, same notation as in Fig. 1.

there are seven coefficients available in expansion (3.3) for A' and six in expansion (3.5) for B, *i.e.* [3/3] and [3/2] approximants are possible, respectively. If, however, the number of zeros in the physical amplitude becomes too large and/or the amplitude becomes too bumpy, it may be impossible for the Padé approximant of a definite order to reproduce all zeros, as a consequence of which poles migrate into the physical region—or, in other words, the Padé approximant does not exist. We observe this behaviour for all amplitudes, though it begins at different energies for all four, depending on their

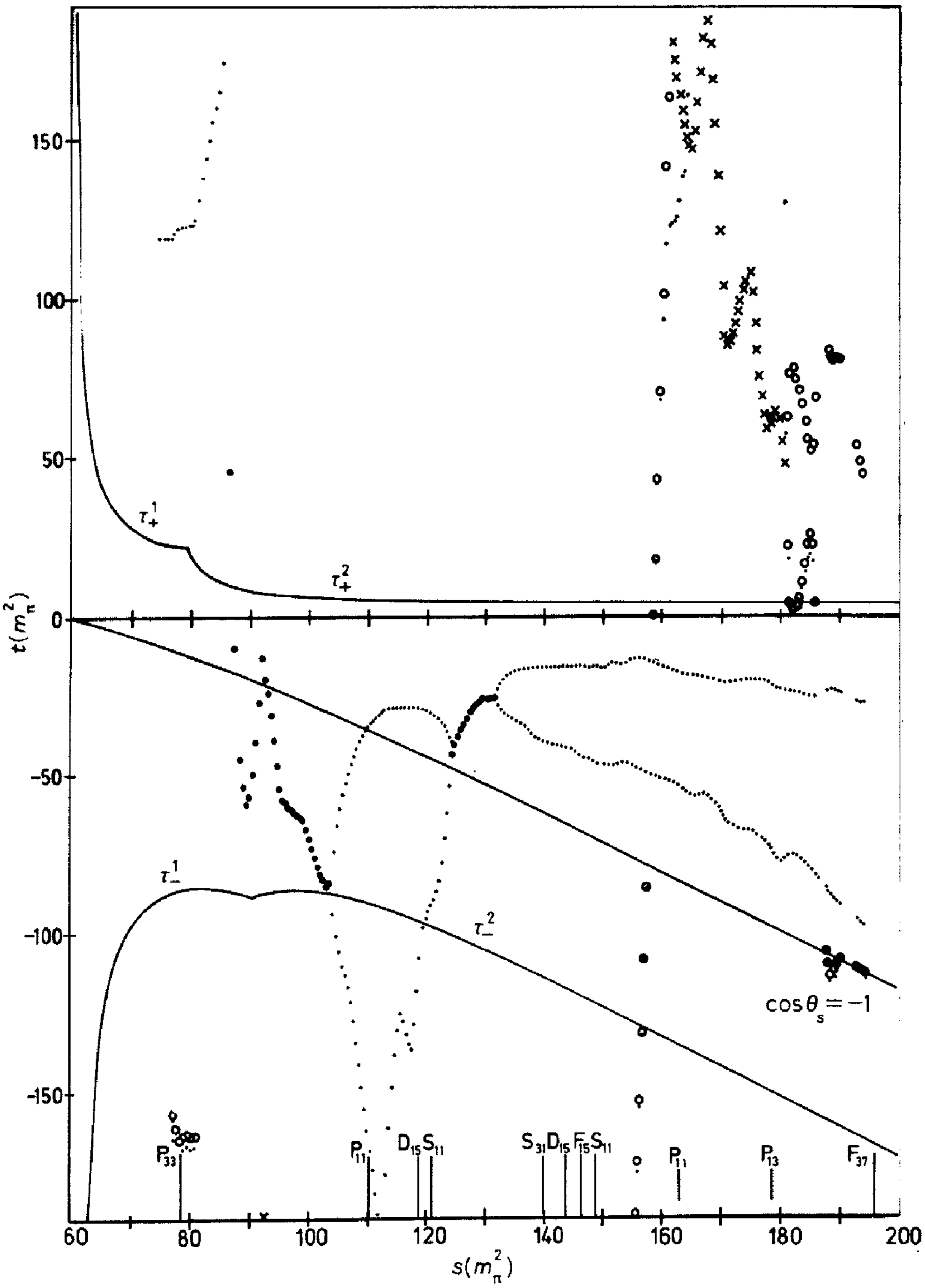

Fig. 4. – Pole-zero distribution for $\operatorname{Im} B^{(-)}$, same notation as in Fig. 1.

smoothness. In order to account for the physical zeros of the amplitudes as well as possible, we chose $[N+1/N]$ approximants in the case that nondiagonal approximants are the highest possible ones.

Discussing now the amplitudes in detail, we observe that $\operatorname{Im} A'^{(+)}$ (Fig. 1) has at higher energies one real and one complex zero in the physical region, the [3/3] Padé approximant existing up to $s \approx 220$. It is interesting to note that the complex zero starts exactly at the position of the $P_{11}(1470)$-resonance. Figure 1 shows that there are no real poles in the region of analyticity. For

$t < 0$ the absence of poles in the shown region of the double-spectral function indicates that the double-spectral function in this region can be assumed to be small. For $t > 0$ we have real and complex poles and doublets.

$\operatorname{Im} A'^{(-)}$ is a much bumpier function (Fig. 2). At higher energies ($s \geqslant 130$) there are three zeros in the physical region and finally the [3/3] Padé approximant is unable to represent the amplitude. At $s \approx 150$ real and complex poles start migrating into the physical region. If one insists on calculating only diagonal or $[N \pm 1/N]$ approximants, only higher-order approximants could fit an amplitude like that. As such higher-order approximants (like a [4/4] *e.g.*) will be very sensitive to the higher partial waves, the Padé ansatz might be a very powerful tool in performing partial-wave analyses in this energy range. In particular one will obtain a smooth extrapolation to the higher partial waves.

At lower energies we observe doublets in the [3/2] approximant at $s \approx 100$. If we calculate the [2/3] approximant, these disappear completely. A real pole appears at $t \leqslant -240$ and a complex pole with real part $\geqslant 120$. This shows that doublets are just instabilities, disappearing with a change of order. It also shows that the double-spectral function in the shown region of $t < 0$ can again be assumed to be small. The complex poles of the [3/3] approximant in the region at $s \approx 125$ have an imaginary part of modulus > 220. In spite of the fact that their real parts are partially in the region of analyticity, their large imaginary parts show that these complex poles must not be considered as disturbing the analyticity properties of the amplitude seriously.

$\operatorname{Im} B^{(+)}$ and $\operatorname{Im} B^{(-)}$ are again smooth functions (Fig. 3 and 4). The only new thing we learn from $\operatorname{Im} B^{(+)}$ is that at $t \approx -100$ there are poles without accompanying zeros in the region of the double-spectral function at $s \approx 90 \div 100$, which are also stable with the order. Figure 3 shows that the pole trajectories of the [1/2] and [2/2] approximants are reasonably well connected. From this we conclude that the double-spectral function $\varrho_{su}(B^{(+)})$ should not be small in this region.

The results for the imaginary parts show that—except for the doublets and the difficulties at higher energies—the poles of the Padé approximants do not occur in the region of analyticity. Thus the Padé approximants for Legendre series seem to probe the singularity structure of functions just as Padé approximants for Taylor series do (see also ref. [9]). The Padé approximants and the original Legendre series deviate from each other only up to 1 % even close to the boundary of the double-spectral function. There is only one exception: for $\operatorname{Im} B^{(+)}$ one has stable poles near the boundary of the double-spectral function ϱ_{su} for $90 \leqslant s \leqslant 100$ and of course then deviations occur.

4.2. *The real parts.* – The boundaries of the analyticity domains of the dispersive parts are much closer to the physical region than those for the absorptive parts. Therefore one expects a different pole distribution for the

Padé approximants. Moreover, the dispersive parts contain the u-channel nucleon pole. The natural thing to do is to subtract the u-pole term from the real part before the calculation of the Padé approximants. The contributions of the u-pole term to the Legendre coefficients $a_l'^{(\pm)}$ and $b_l^{(\pm)}$ are

$$a'^{(\pm)}_{l,\text{pole}} = \pm(2l+1)\frac{2M^3f^2}{q^2}(s-M^2)Q_l(z)\,, \tag{4.1}$$

$$b^{(\pm)}_{l,\text{pole}} = \pm(2l+1)\frac{2M^2f^2}{q^2}Q_l(z)\,, \tag{4.2}$$

where f^2 is the pion-nucleon coupling constant,

$$z = 1-\frac{s-M^2-2}{2q^2}\,, \tag{4.3}$$

and $Q_l(z)$ is the Legendre function of the second kind. For amplitudes which are calculated from the Lovelace-Almehed phase-shift analysis it is, however, not clear which value of f^2 one should choose to subtract the u-pole term.

Instead of those amplitudes we have investigated two sets of amplitudes with known f^2:

i) Amplitudes obtained from fixed-t dispersion relations

$$\operatorname{Re}X^{(\pm)}(\nu,t) = X^{(\pm)}_{\mathcal{N}}(\nu,t)+\frac{P}{\pi}\int\limits_{1+t/4M}^{\infty}\mathrm{d}\nu'\,\operatorname{Im}X^{(\pm)}(\nu',t)\left[\frac{1}{\nu'-\nu}\pm\frac{1}{\nu'+\nu}\right], \tag{4.4}$$

$$\nu=\frac{s-u}{4M}\,, \tag{4.5}$$

$$\operatorname{Re}\tilde{X}^{(\pm)}(\nu,t) = \operatorname{Re}X^{(\pm)}(\nu,t)-X^{(\pm)}_{\mathcal{N},u\text{-pole}}(\nu,t)\,, \tag{4.6}$$

where $X^{(\pm)}$ is one of the amplitudes $A'^{(-)}$, $B^{(\mp)}$, and $X^{(\pm)}_{\mathcal{N}}$ the nucleon pole term. The relation for $A'^{(+)}$ has to be subtracted or to be changed in a finite-contour dispersion relation. Both methods yield the same $\operatorname{Re}A'^{(+)}$ (see ref. [10]). For the numerical evaluation of eq. (4.4) we used the phase-shift analysis [5] as far as possible; at higher energies the imaginary parts were calculated from the Regge model of BARGER and PHILLIPS [11]. Of course, the amplitudes obtained from eq. (4.4) are *a priori* analytic functions of t in the range given by (3.12). Dispersion relations are in this respect analyticity laboratories.

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[10] J. BAACKE and J. ENGELS: *Nucl. Phys.*, **51** B, 434 (1973).
[11] V. BARGER and R. J. N. PHILLIPS: *Phys. Rev.*, **187**, 2210 (1969).

ii) Amplitudes calculated from phase shifts given by NIELSEN and OADES ([6]) in the low-energy region (pion laboratory kinetic energy $T_\pi \leqslant 270$ MeV). Their work is based on finite-contour dispersion relations but not identical to our way i) of obtaining real parts of amplitudes. We have used the phase shifts from their Table 2, *i.e.* with $f^2 = 0.081$, Coulomb corrections and Lovelace-Almehed phase shifts above $T_\pi = 270$ MeV as input.

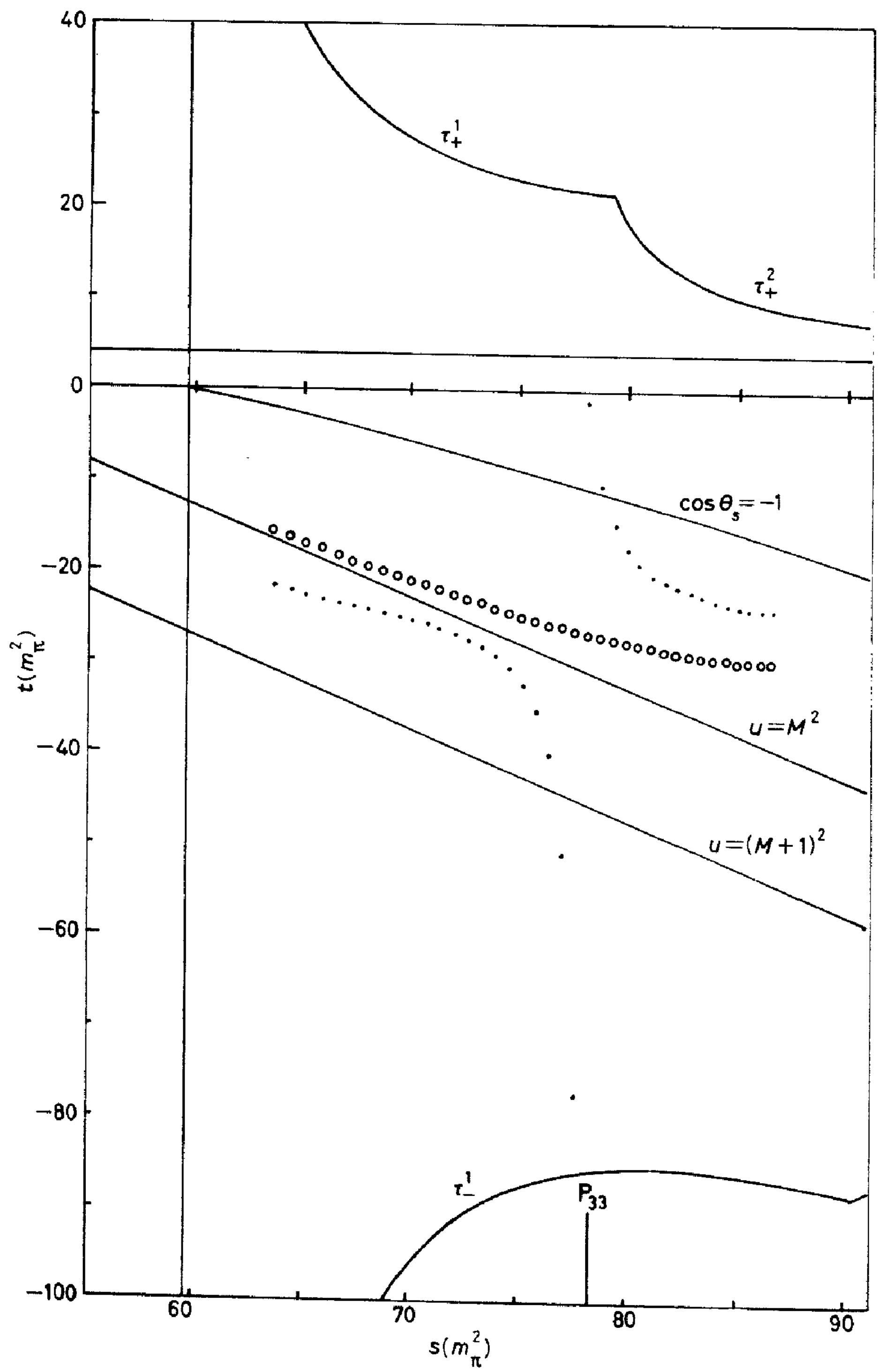

Fig. 5. – Distribution of poles (∘) and zeros (·) of the [2/1] approximants for Re $B^{(+)}$ obtained from eq. (3.4).

Both methods are limited to the low-energy region, because the fixed-t dispersion relations may only be used for $t > -26$, if one uses a Legendre series to calculate $\operatorname{Im} X^{(\pm)}$. To demonstrate the influence of the u-pole term we have calculated the [2/1] Padé approximants of $\operatorname{Re} B^{(+)}$. The location of the poles in the Mandelstam plane along the line $u = M^2$ (see Fig. 5) shows

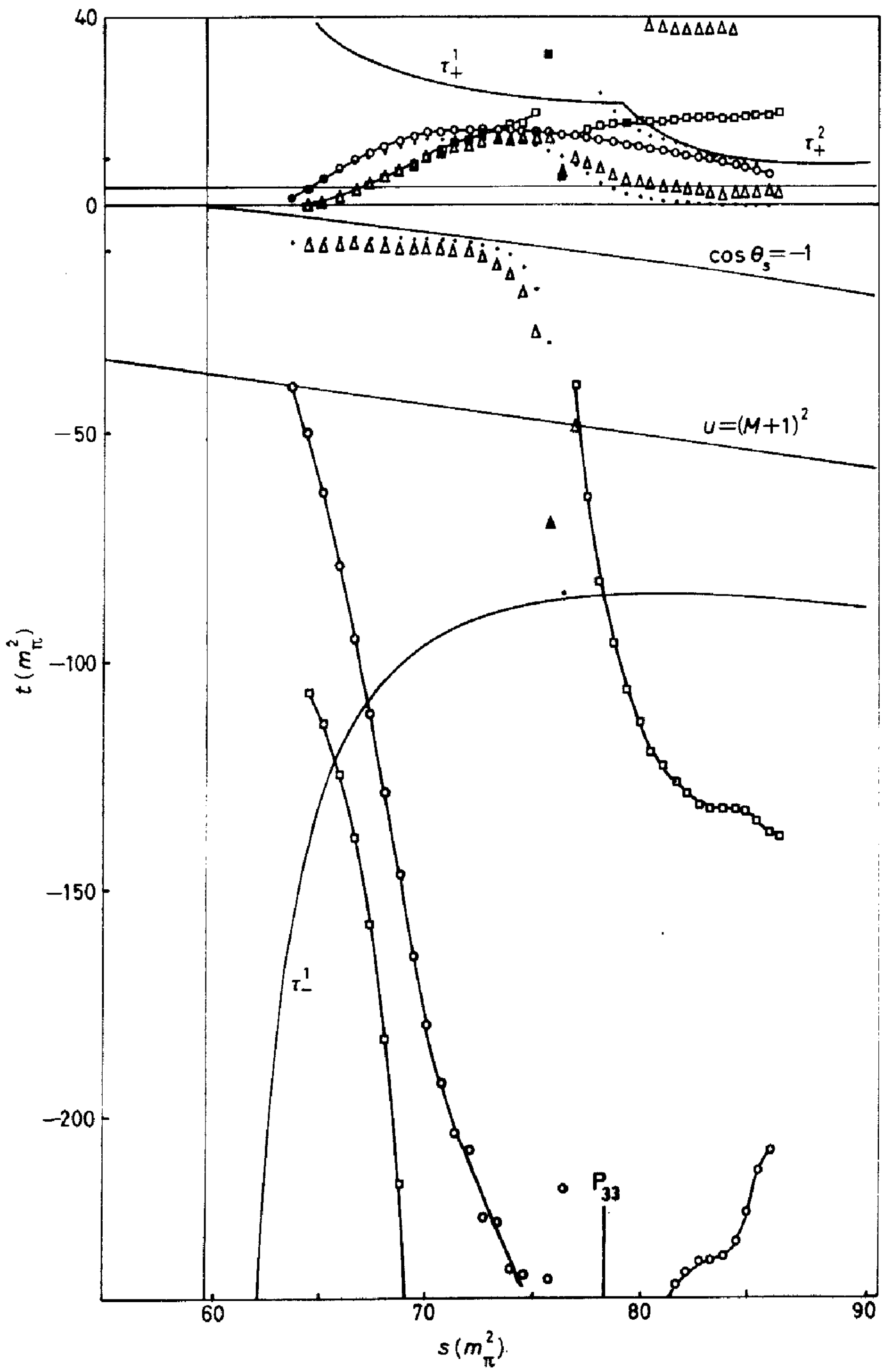

Fig. 6. – Pole-zero distribution of [2/2] approximants for $\operatorname{Re} A'^{(\pm)}$, obtained from eqs. (4.4) and (4.6). We use the following symbols: ○, •, × and ● for the isospin-even amplitudes and □, △, ■ and ▲ for the isospin-odd amplitudes for real pole, real zero, real part of complex pole and real part of complex zero, respectively.

that this amplitude is dominated by the u-pole term. If we do not subtract this known effect in an amplitude, the poles will in general show up in the region of analyticity. For the amplitudes $\mathrm{Re}\,\tilde{X}^{(\pm)}(\nu, t)$, however, we find the expected picture (see Fig. 6 and 7): there are no poles in the region of analyticity. One can even identify the remarkably constant string of poles around $t \approx 23 \div 24$ for $\mathrm{Re}\,B^{(-)}$ as an effective ρ-pole (Fig. 7). Remember $m_\rho^2 = 30$. It is clear

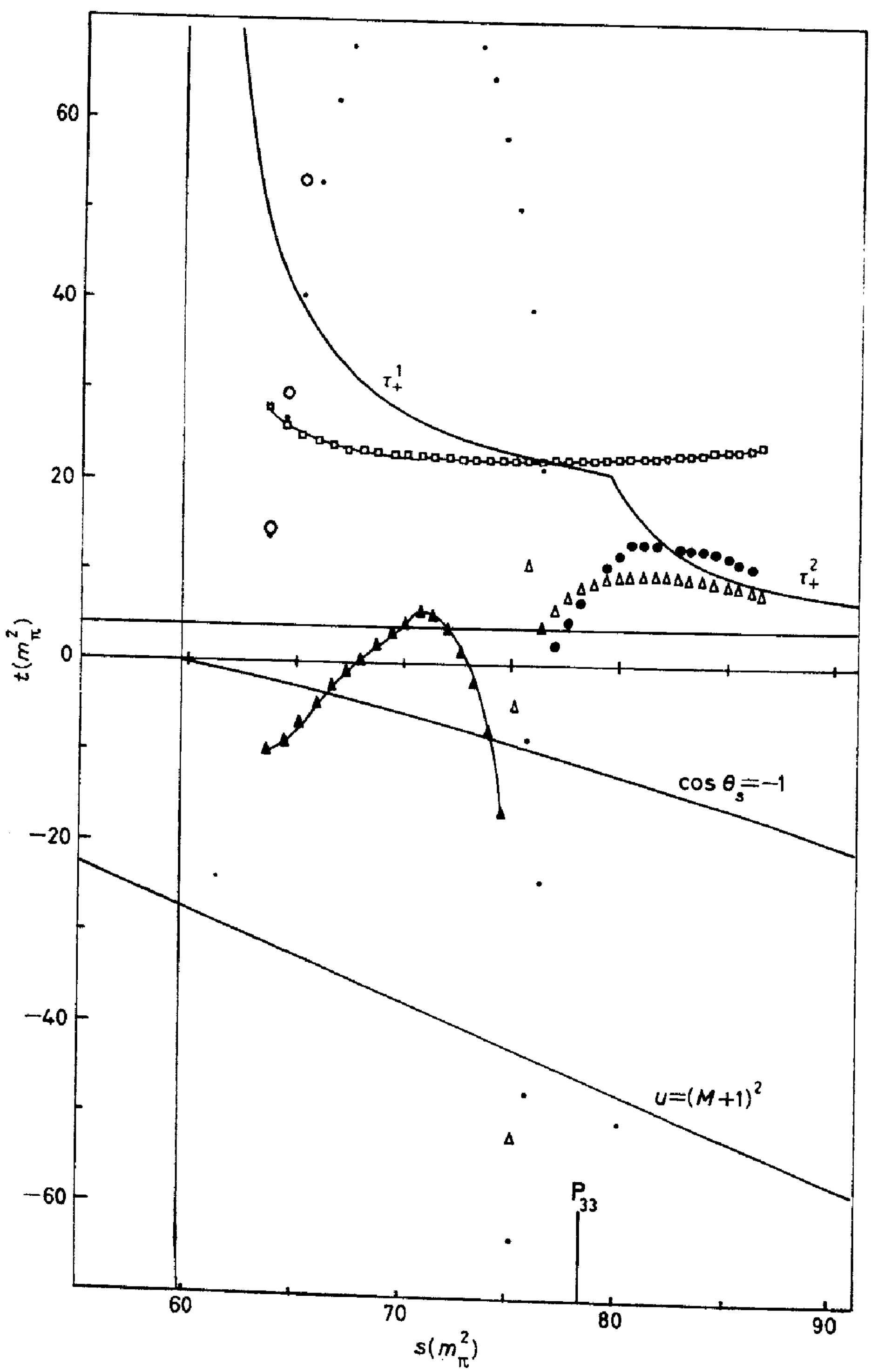

Fig. 7. – Pole-zero distribution for the [2/1] approximants of $\mathrm{Re}\,B^{(\pm)}$, obtained from eqs. (4.4) and (4.6), same notation as in Fig. 6.

why we have this effect. The Regge contribution to $\mathrm{Re}\,B^{(-)}$ in eq. (4.4) contains the ρ trajectory and though this contribution is only about 10 % of the whole amplitude the Padé approximant recognizes its pole structure. We do not have the same effect in $\mathrm{Re}\,A'^{(-)}$ (Fig. 6), because there the ρ-Regge contribution is only about 1 % of the amplitude in the low-energy region. Near

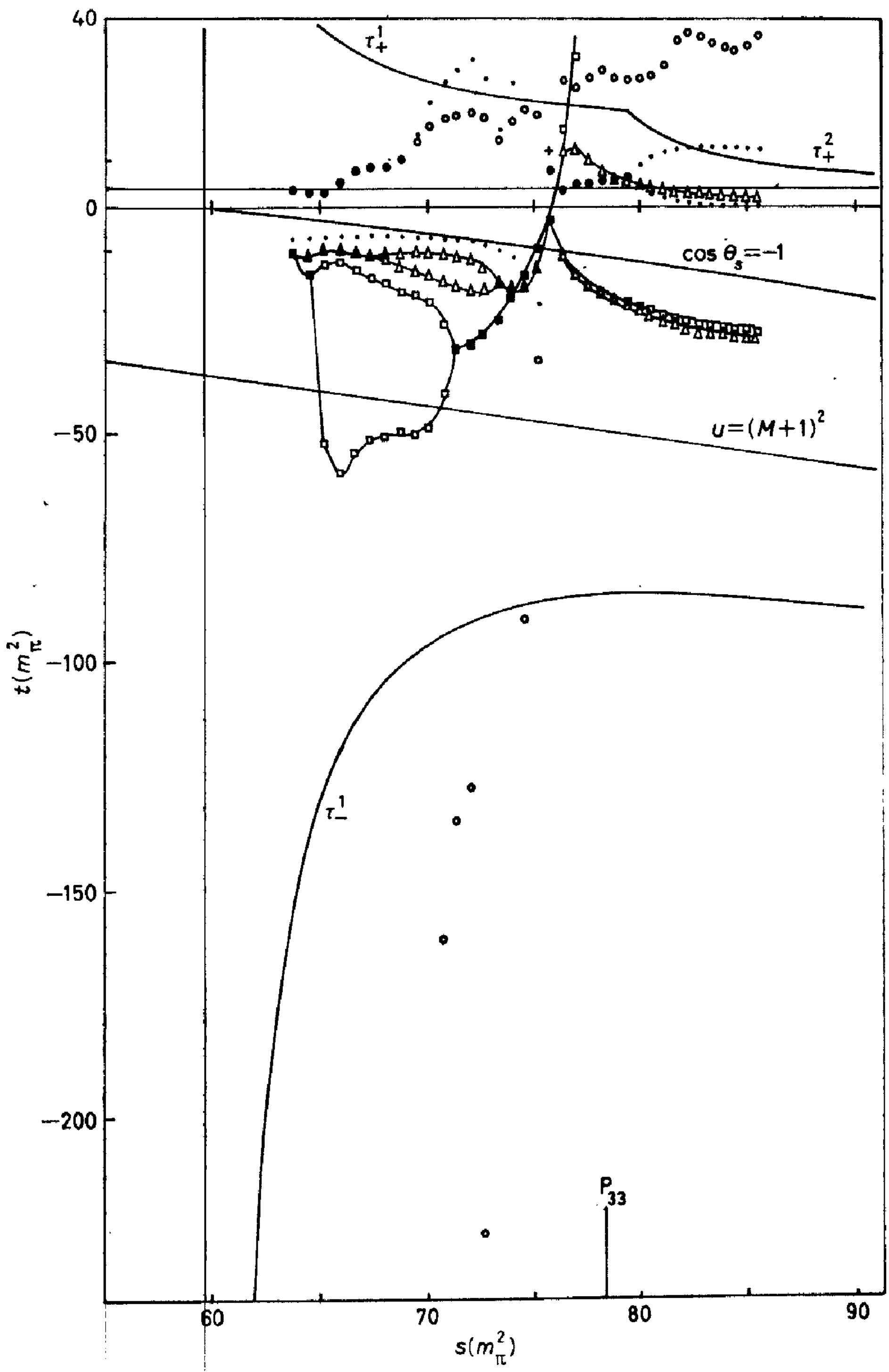

Fig. 8. – Pole-zero distribution of the [2/2] approximants for $\mathrm{Re}\,A'^{(\pm)}$, eq. (4.6), where the real part was calculated from the phase shifts of NIELSEN and OADES, same notation as in Fig. 6.

the position of the P_{33}-resonance all real parts show rapid movements of the zeros, a behaviour which was of course not found for the imaginary parts.

Finally we show in Fig. 8 and 9 the pole-zero distributions of the set ii) of amplitudes. Whereas the isospin-even amplitudes have—as expected—no poles in the region of analyticity, their isospin-odd counterparts seem not to have the right analytic properties. This effect is extremely strong in $\mathrm{Re}\,B^{(-)}$.

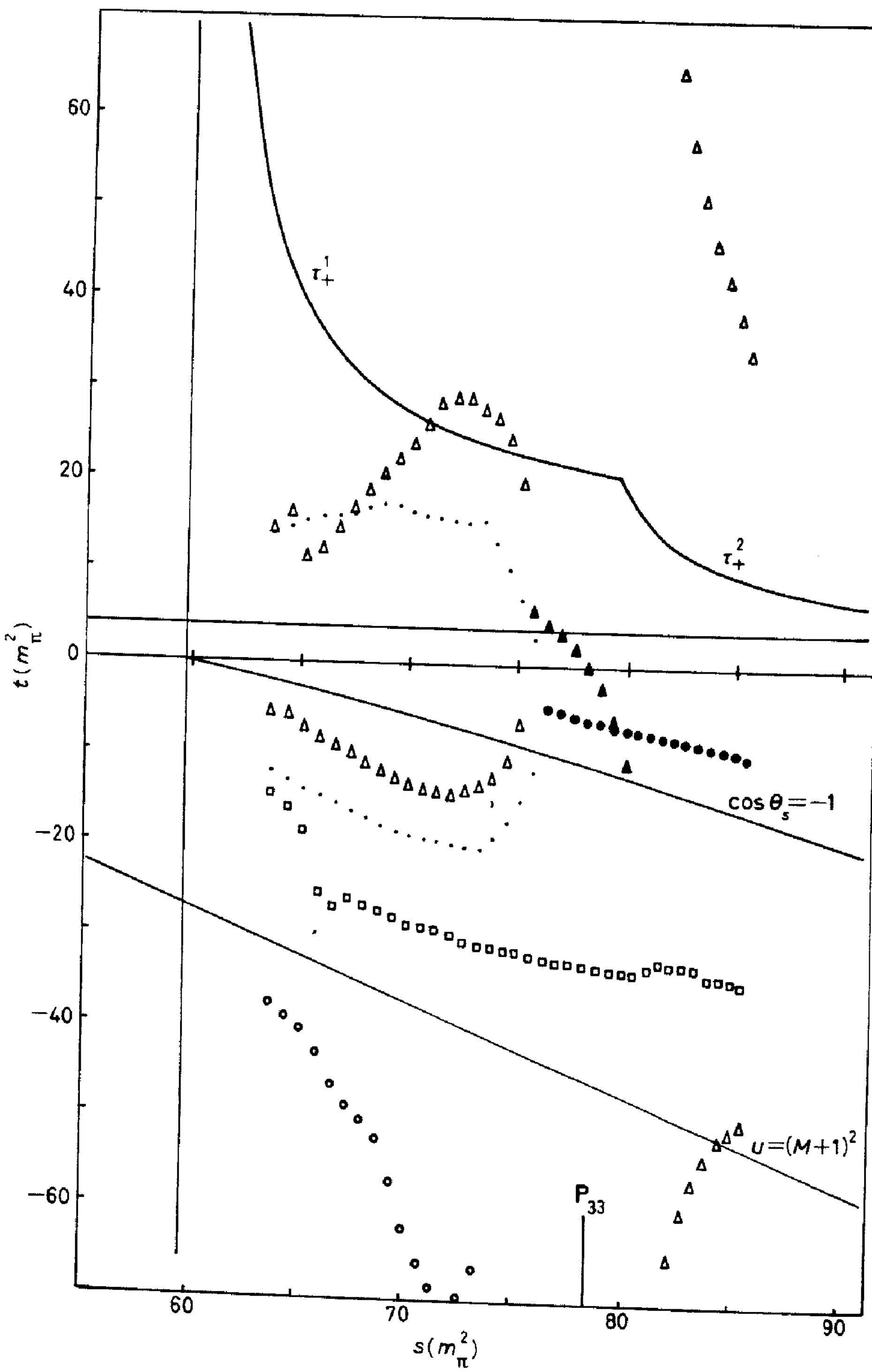

Fig. 9. – Pole-zero distribution of the [2/1] approximants for $\mathrm{Re}\,B^{(\pm)}$, eq. (4.6), where the real part was calculated from the phase shifts of NIELSEN and OADES, same notation as in Fig. 6.

5. – Conclusion.

We have applied Padé approximants for Legendre series to $\pi\mathcal{N}$ scattering. The Padé approximants are expected to probe the singularity structure of the scattering amplitudes.

We find in general that

i) there are no poles of Padé approximants in the region of analyticity of an amplitude, except for pole-zero pairs, which are unstable with the order of Padé approximants;

ii) poles without accompanying zeros on the cuts are approximately stable and thus seem to indicate important regions of the cuts.

The imaginary parts of the amplitudes $A'^{(\pm)}$ and $B^{(\pm)}$ were calculated from the Almehed-Lovelace phase shifts [5]. We observe that

iii) for negative t, between the boundaries of the physical region and the region of the double-spectral functions, the Padé approximants and the original Legendre sums deviate only up to 1 % from each other. This is not true for $\operatorname{Im} B^{(+)}$ at $s \approx 90 \div 100$.

For the real parts without the u-pole terms from fixed-t dispersion relations we arrive at the following results:

iv) as expected, the poles are closer to the physical region but still on the cut;

v) the ρ-Regge trajectory, which is inserted via the fixed-t dispersion relation into the isospin-odd amplitudes, is rediscovered for $\operatorname{Re} B^{(-)}$ as an effective ρ-pole of the Padé approximant at $t \approx 24$. For $\operatorname{Re} A'^{(-)}$ the Regge contribution is too small to have this effect.

The real parts without the u-pole terms as calculated from the phase shifts of NIELSEN and OADES [6] behave similarly in the case of the (+)-amplitudes, but for the (−)-amplitudes one obtains poles in the analyticity region.

As a consequence of our result iii) one may use in applications the Legendre sums for the imaginary parts to lower t values than -26. This might especially be useful for the calculation of the t-channel partial waves (see, *e.g.*, ref. [12]). Padé approximants might be helpful during phase-shift analyses, since they discriminate between more or less analytic solutions. If we start, however, directly with a Padé ansatz in phase-shift analyses, it seems to be difficult to build in unitarity.

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[12] G. HÖHLER, R. STRAUSS and H. WUNDER: Karlsruhe preprint (1968), unpublished; W. R. FRAZER and J. R. FULCO: *Phys. Rev.*, **117**, 1609 (1960).

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● RIASSUNTO (*)

Ci si attende che gli approssimanti di Padé sondino la struttura della singolarità delle ampiezze di scattering. Si applicano gli approssimanti di Padé delle serie di Legendre alle ampiezze $\pi\mathcal{N}$, che si ottengono dagli spostamenti di fase. Si sono calcolate le parti immaginarie dall'analisi degli spostamenti di fase di Almehed e Lovelace, e le parti reali direttamente dalle relazioni di dispersione a t fisso o dall'analisi degli spostamenti di fase di Nielsen e Oades. Per tutte le ampiezze (eccetto quelle di isospin dispari di Nielsen-Oades) i poli degli approssimanti di Padé giacciono fuori della regione di analicità nel piano di Mandelstam. Gli approssimanti di Padé e le serie di Legendre troncate per le parti immaginarie deviano fra loro solo sino all'1% anche presso il contorno della funzione spettrale doppia (salvo per $B^{(+)}$).

(*) *Traduzione a cura della Redazione.*

Падэ аппроксимации для ряда Лежандра в $\pi\mathcal{N}$ рассеянии.

Резюме (*). — Предполагается, что Падэ аппроксимации позволят исследовать структуру сингулярностей амплитуд рассеяния. Мы применяем Падэ аппроксимации для ряда Лежандра для $\pi\mathcal{N}$ амплитуд, которые получены из анализа фазовых сдвигов. Вычисляются мнимые части из анализа фазовых сдвигов Алмехеда и Лавлейса. Формулы для вещественных частей получаются непосредственно из дисперсионных соотношений при фиксированном t и из анализа фазовых сдвигов Нильсена и Оадса. Для всех амплитуд (за исключением изоспиновых нечетных Нильсена-Оадса) полюса Падэ аппроксимаций расположены вне области аналитичности в плоскости Манделстама. Падэ аппроксимации и обрезанный ряд Лежандра для мнимых частей отличаются друг от друга только на 1% даже вблизи границы двойний спектральной функции (за исключением $B^{(+)}$).

(*) *Переведено редакцией.*