On a Generalized Matching Problem Arising in Estimating the Eigenvalue Variation of Two Matrices

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It is shown that if $G$ is a graph having vertices $P_1, P_2, \ldots, P_n, Q_1, Q_2, \ldots, Q_n$ and satisfying some conditions, then there is a permutation $\sigma$ of $\{1, 2, \ldots, n\}$ such that there is a path, for $i = 1, 2, \ldots, n$ connecting $P_i$ with $Q_{\sigma(i)}$, having a length at most $(n/2)$. This is used to prove a theorem having applications in eigenvalue variation estimation.

For complex $n \times n$-matrices $A$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ and $B$ with eigenvalues $\mu_1, \ldots, \mu_n$, it is possible to give bounds for the "spectral-variation" $S_A(B) = \max_i \min_j |\lambda_i - \mu_j|$, depending only on $\|A\|, \|B\|$ and $\|A - B\|$. Here $\| \|$ denotes the spectral-norm (e.g. [1]). These bounds are also bounds on

$$\delta = \max_{0 \leq t \leq 1} \max \{S_A(tB + (1-t)A), S_B(tB + (1-t)A)\}.$$

It follows from a continuity argument that each connected component of $\bigcup_{i=1}^{n} \{z : |z - \mu_i| \leq \delta\}$ and of $\bigcup_{i=1}^{n} \{z : |z - \lambda_i| \leq \delta\}$ contains as many eigenvalues of $A$ as of $B$. One is in fact interested in the "eigenvalue variation"

$$\nu(A, B) = \min_{\sigma} \max_i |\lambda_i - \mu_{\sigma(i)}|,$$

where $\sigma$ runs through all permutations of $\{1, 2, \ldots, n\}$. It is easy to see that $\nu(A, B) \leq (2n - 1)\delta$. It was suspected that $2n - 1$ can be replaced by $n$ for $n$ odd and $n - 1$ for $n$ even. Hence the question arose whether the following statement is true.

**STATEMENT 1.** Let $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and $\mu = \{\mu_1, \mu_2, \ldots, \mu_n\}$ be two sets of not necessarily distinct points in the complex plane. Suppose that for every connected component $D$ of the domain $\bigcup_{i=1}^{n} \{z : |z - \mu_i| \leq 1\}$ or of the domain $\bigcup_{i=1}^{n} \{z : |z - \lambda_i| \leq 1\}$ the number of elements of $\lambda$ contained in $D$ equals the number of elements of $\mu$ contained in $D$. Then there is a permutation $\sigma$ of $\{1, 2, \ldots, n\}$ such that for $i = 1, 2, \ldots, n$,

$$|\lambda_i - \mu_{\sigma(i)}| \leq \begin{cases} n & \text{for } n \text{ odd}, \\ n - 1 & \text{for } n \text{ even}. \end{cases}$$

Since we shall answer the above question in the affirmative, we will refer to Statement 1 as Theorem 1.

It turns out that a much more general result is true. It will be formulated as Theorem 2 and proved below in graph-theoretical terms.

If $A$ and $B$ are vertices in a connected graph, then we shall use the notation $L(AB)$ for a path with endpoints $A, B$ and $l(AB)$ for its length, i.e. the number of the edges in it or, if the edges are weighted, the sum of the weights of its edges. As usual the distance $d(AB)$ means the length of the shortest path connecting $A, B$.

Denote by $\{m\}$ the least integer not smaller than $m$.

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We define now a class \( \Gamma_n \) of graphs. A graph \( G \) will belong to \( \Gamma_n \) if it has the following structure:

(i) The vertex set of \( G \) is the union of two disjoint sets \( V_p \) and \( V_q \), each containing exactly \( n \) elements.

(ii) Let \( G_p \) and \( G_q \) be the induced subgraphs of \( G \) on the sets \( V_p \) and \( V_q \). Let \( B_{pq} \) be the induced bipartite subgraph with cells \( V_p \) and \( V_q \). Then the following condition holds for each connected component \( D \) of \( G_p \): the number of vertices in \( V_q \) joined by an edge to some vertex in \( D \) equals the number of vertices in \( D \). A corresponding condition holds for every connected component of \( G_q \).

(iii) Edges \( B_{pq} \) stemming out from the same vertex of \( V_p \) have the other endpoint in the same connected component of \( G_q \), and vice versa interchanging \( p \) with \( q \).

Notice that from (i), (ii), (iii) it follows

(iv) The degree of each vertex in \( B_{pq} \) is at least 1.

Actually (i), (ii), (iii) and (i), (ii), (iv) are equivalent.

A path connecting a vertex \( P \in V_p \) with a vertex \( Q \in V_q \) will be said to be proper if it contains exactly one edge of \( P_{pq} \), this edge has at least one of the vertices \( P, Q \) as endpoints, and

\[
|PQ| < \left\lfloor \frac{n}{2} \right\rfloor.
\]

**THEOREM 2.** If \( G \) is a member of \( \Gamma_n \), and \( V_p = \{P_1, P_2, \ldots, P_n\} \), \( V_q = \{Q_1, \ldots, Q_n\} \), then there is a permutation \( \sigma \) of \( \{1, 2, \ldots, n\} \) such that for each \( i = 1, 2, \ldots, n \) there is a proper path \( L(P_iQ_{\sigma(i)}) \).

**PROOF.** Let \( A_i \) be the subset of \( V_q \) such that if \( Q \in A_i \) then there is a proper path \( L(P_iQ) \). The set \( A_i \) is non-empty for \( i = 1, 2, \ldots, n \) by property (iv). We will prove Theorem 2 by showing that the sets \( A_1, A_2, \ldots, A_n \) have a system of distinct representatives. This will be done by verifying Hall's condition [2]. Thus, we shall verify the condition:

\[
\left| \bigcup_{i=1}^{k} A_i \right| \geq k, \quad k = 1, 2, \ldots, n; \quad \{i_j\}_{j=1}^{k} \subset \{1, 2, \ldots, n\}.
\]

Let \( s \geq 1 \) be such that \( G_q^1, G_q^2, \ldots, G_q^s \) are the connected components of \( G_q \) and let \( m_1, m_2, \ldots, m_s \) be the cardinalities of the corresponding vertex sets \( V_q^1, V_q^2, \ldots, V_q^s \).

Notice that if

\[
m_i = \left\lfloor \frac{n}{2} \right\rfloor
\]

and there is an edge from \( P_i \) to a vertex of \( G_q^j \), then \( |A_i| \geq m_j \), and notice that (2) holds for all but possibly one value of \( j \). Choose the notation so that \( m_s \geq m_j \) for \( j = 1, 2, \ldots, s - 1 \).

Consider the set \( R \) of vertices \( P_{i_1}, P_{i_2}, \ldots, P_{i_k} \) and the corresponding sets \( A_{i_1}, A_{i_2}, \ldots, A_{i_k} \).

**Case 1.** Either \( m_s \leq \{n/2\} \) or there is no edge from \( P_{i_j}, j = 1, 2, \ldots, k, \) to \( G_q^s \).

We show that in either case (1) holds. Indeed, since each \( A_{i_j} \) contains at least one of the components of \( G_q^s \), one can group together equal components and get:

\[
\left| \bigcup_{j=1}^{k} A_{i_j} \right| \geq \sum_{\nu=1}^{k} \left| V_q^\nu \right| = \sum_{\nu=1}^{k} \left| V_q^\nu \right| \sum_{\nu=1}^{k} m_{i_\nu}.
\]
On the other hand $k$ is less than or equal to the total number of vertices $P$ such that there is an edge from $P$ to one of the sets $V_{q}$, but this number is smaller or equal, by property (ii) of graphs $\Gamma_{n}$, to $\sum_{i=1}^{h} m_{c_{i}}$.

Case 2. $m_{c} \geq \{n/2\} + 1$ and there is an edge from the set $R$ to some vertex of $V_{q}$.

In this case clearly (1) holds provided $k \leq \{n/2\}$; indeed, for some $i$, $|A_{i}| \geq \{n/2\}$. We claim that (1) holds even if $k > \{n/2\}$.

Suppose the contrary

$$|\bigcup_{i=1}^{k} A_{i}| < k.$$ 

It follows that there are at least $n - k + 1$ elements $Q$ in $V_{q}$ which are not in $\bigcup_{i=1}^{k} A_{i}$. But $k > \{n/2\}$ implies $n - k + 1 \leq \{n/2\}$.

Define $B_{i}$, $i = 1, 2, \ldots, n$, to be the set of elements $P$ of $V_{p}$ for which there is a proper path $L(QP)$. We have shown that (1) holds for $k \leq \{n/2\}$. Hence, by symmetry, $\bigcup_{i=1}^{l} B_{i}$, for $l \leq \{n/2\}$, in particular when $l = n - k + 1$. Therefore, for at least one of the $n - k + 1$ considered vertices $Q$ there is a proper path $L(QP)$ where $P \in R$, a contradiction, since if $L(QP)$ is proper, then $L(PQ)$ is also proper.

**Corollary 1.** If $G$ is as in Theorem 2, then there is a permutation $\sigma$ of $\{1, 2, \ldots, n\}$ such that $d(P_{Q_{\sigma(i)}}) \leq \{n/2\}$.

**Corollary 2.** If $G$ is as in Theorem 2 and if weight 1 is assigned to every edge in $B_{pq}$ and weight 2 to every edge in $G_{p}$ and $G_{q}$, then there is a permutation $\sigma$ of $\{1, 2, \ldots, n\}$ such that

$$d(P_{Q_{\sigma(i)}}) \leq \begin{cases} n & \text{for } n \text{ odd}, \\ n-1 & \text{for } n \text{ even}. \end{cases}$$

(3)

We shall omit the proofs.

**Proof of Theorem 1.** Given the set of points $\lambda = \{\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}\}$ and $\mu = \{\mu_{1}, \mu_{2}, \ldots, \mu_{n}\}$, consider the graph $G$ having vertex set $\lambda \cup \mu$. Putting $\lambda = V_{p}$, $\mu = V_{q}$, two vertices both in $V_{p}$ or both in $V_{q}$ are joined by an edge if the distance between them is at most 2. Two vertices, one in $V_{p}$ and one in $V_{q}$, are joined by an edge if the distance between them is at most 1. The graph $G$ is clearly a member of $\Gamma_{n}$. Assigning weights as in Corollary 2, condition (3) follows and this implies Theorem 1.

It seems to be of interest to formulate a particularization of Theorem 2.

**Theorem 3.** Suppose $G$ is a graph, the vertex set of which consists of the union of two disjoint sets $V_{p} = \{P_{1}, P_{2}, \ldots, P_{n}\}$ and $V_{q} = \{Q_{1}, Q_{2}, \ldots, Q_{n}\}$, and the edge set of which satisfies the following two conditions.

(i) The induced subgraphs on $V_{p}$ and $V_{q}$ are connected.

(ii) Each vertex of the bipartite graph induced on $V_{p}$ and $V_{q}$ as cells has degree at least 1.

Then there is a permutation $\sigma$ of $\{1, 2, \ldots, n\}$ such that there is a proper path $L(P_{Q_{\sigma(i)}})$ for each $i = 1, 2, \ldots, n$.

**Remark.** Theorem 2 is sharp, i.e. for every $n$ there are graphs in $\Gamma_{n}$ for which it is impossible to choose in the definition of proper paths a shorter length than given in (0).
As an example for odd \( n \) consider the graph of Figure 1. This graph is a member of \( T_{2k+1} \). It even satisfies the assumptions of Theorem 3. But clearly for the \( k+1 \) vertices on top of \( V_p \) only \( k \) vertices of \( V_q \) can be closer than required by condition (0).

This situation can occur in the case of Theorem 1 also, when all the points \( \lambda \) and \( \mu \) are on the real line. If \( k = 2 \), for instance, let the points of \( \lambda \) and of \( \mu \) be the points having abscissas

\[
\lambda_1 = \lambda_2 = \lambda_3 = 0, \quad \lambda_4 = 2, \quad \lambda_5 = 4, \quad \mu_1 = 1, \quad \mu_2 = 3, \quad \mu_3 = \mu_4 = \mu_5 = 5.
\]

This shows that the condition in Theorem 1 is also sharp.

**References**


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