QF – 1 RINGS OF GLOBAL DIMENSION ≤ 2

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R. M. Thrall [10] introduced QF – 1, QF – 2 and QF – 3 rings as generalizations of quasi-Frobenius rings. (For definitions, see section 1. It should be noted that all rings considered are assumed to be left and right artinian.) He proved that QF – 2 rings are QF – 3 and asked whether all QF – 1 rings are QF – 2, or, at least, QF – 3. In [9] we have shown that QF – 1 rings are very similar to QF – 3 rings. On the other hand, K. Morita [6] gave two examples of QF – 1 rings, one of them not QF – 2 and therefore not QF – 3, the other one QF – 3, but not QF – 2. The global dimension of the latter ring is 2, and the following theorem shows that under this assumption a QF – 1 ring must always be QF – 3.

THEOREM. A QF – 1 ring of left global dimension ≤ 2 is a QF – 3 ring.

In order to classify finite dimensional algebras, T. Nakayama [8] defined the dominant dimension dom dim R of a ring R. Since dom dim R ≥ 1 if and only if R is a QF – 3 ring, and, in this case, dom dim R ≥ 2 if and only if the minimal faithful left R-module is balanced, we may reformulate the theorem as follows: a QF – 1 ring R of left global dimension ≤ 2 has dom dim R ≥ 2. It was proved by K. R. Fuller [4] that for a ring R with dom dim R ≥ 2, every faithful module which is either projective or injective has to be balanced. Naturally, the question arises whether it is possible to characterize those rings R of left global dimension ≤ 2 which have dom dim ≥ 2 by the fact that certain faithful R-modules are balanced. This question seems to be interesting in view of the importance of the class of rings of global dimension ≤ 2 and dominant dimension ≥ 2, recently demonstrated by M. Auslander [1].

The proof of the theorem uses besides the socle conditions of [9] a result concerning the right socle of a QF – 1 ring, and the methods to prove this are similar to those developed in [9]. The assumption in the theorem on the global dimension can be replaced by the (weaker) condition that the right socle, considered as a left module, is projective.

1. Preliminaries. Throughout the paper, R denotes a (left and right) artinian ring with unity. By an R-module we understand a unital R-module and the symbols RM and MR will be used to underline the fact that M is a left or a right R-module, respectively.

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The length of the module $M$ will be denoted by $\partial M$. For every module $M$, \( \text{Rad} M \) is the intersection of all maximal submodules. The radical of $R$ is by definition $\text{Rad}_R R$; it will be denoted by $W$. It is well-known that for an artinian ring, $W$ is nilpotent. The submodule of $M$ generated by all simple submodules, is called the socle, $\text{Soc} M$ of $M$. Since $R$ is artinian, we have for every left $R$-module, $\text{Rad} M = WM$ and $\text{Soc} M = \{ m \in M | Wm = 0 \}$. Considering $R_R$, we get the left socle $L = \text{Soc} R_R$, considering $R_R$, we get the right socle $J = \text{Soc} R_R$ of $R$.

If $e$ is an idempotent, $Re$ always will be considered as a left $R$-module, and the $R$-homomorphisms $Re \to Re'$ (where $e'$ is another idempotent) will be identified with the elements of $eRe'$. Also, it should be noted that $Re$ and $Re'$ are isomorphic if there are elements $x \in eRe'$ and $y \in e'Re$ with $exy = e$. The ring $R$ is called a basis ring if for orthogonal idempotents $e$ and $e'$, $Re$ and $Re'$ never are isomorphic. Basis rings can be characterized by the fact that $eR(1 - e) \subseteq W$ for every idempotent $e$. If $R$ is an arbitrary artinian ring and we write

$$1 = \sum_{i,j} e_{ij}$$

with primitive and orthogonal idempotents $e_{ij}$ such that $Re_{ij} = Re_{k,l}$ if and only if $i = k$, then, for $E = \sum_i e_{ii}$, the ring $ERE$ is a basis ring which is Morita equivalent to $R$.

The ring $R$ is a $QF - 3$ ring if $R$ has a unique minimal faithful left $R$-module $R_X$ (that is, $R_X$ is faithful, and is a direct summand of every faithful left $R$-module). A $QF - 3$ ring also has a unique minimal faithful right $R$-module. The ring $R$ is $QF - 3$ if and only if for every primitive idempotent $e$ with $Je \neq 0$, the socle $Le$ of $Re$ is simple, and similarly for every primitive idempotent $f$ with $fL \neq 0$, the socle $fJ$ of $fR$ is simple [2, Theorem (3.6)].

Module homomorphisms always act from the opposite side as the operators; in particular, every left $R$-module $R_M$ defines a $\mathcal{C}$-module $M_{\mathcal{C}}$, where $\mathcal{C}$ is the centralizer of $R_M$. The double centralizer $\mathcal{D}$ of $R_M$ is the centralizer of $M_{\mathcal{C}}$, and there is a canonical ring homomorphism $R \to \mathcal{D}$. The module $R_M$ is called balanced if this morphism $R \to \mathcal{D}$ is surjective. If every finitely generated faithful (left or right) $R$-module is balanced, then $R$ is said to be a $QF - 1$ ring. Until now, no internal characterization of $QF - 1$ rings seems to be known, but in [9] certain necessary socle conditions were proved. For the convenience of the reader and for later reference, we recall these conditions: If $R$ is a $QF - 1$ ring and $e$ and $f$ are primitive idempotents with $f(L \cap J)e \neq 0$, then

1. either $\partial Re = 1$ or $\partial fL R = 1$,
2. we have $\partial Re \times \partial fJ R \leq 2$,
3. $\partial Re = 2$ implies $Je \subseteq Le$, and
4. $\partial fJ R = 2$ implies $fL \subseteq fJ$.

In particular, (2) shows that a $QF - 1$ ring is very similar to a $QF - 3$ ring. If $R_M$ is an indecomposable module of finite length, then the centralizer $\mathcal{C}$
of \( M \) is a local ring. Consequently, all simple \( \mathcal{C} \)-modules are isomorphic. Moreover, the radical \( \mathcal{W} \) of \( \mathcal{C} \) is nilpotent, thus the radical of \( M_{\mathcal{C}} \) is a proper submodule, and \( \text{Soc} M_{\mathcal{C}} \) is essential in \( M_{\mathcal{C}} \). If \( rM \) and \( rN \) are modules, then elements in the double centralizer of \( r(M \oplus N) \) can be constructed as follows: Let \( \mathcal{C} \) be the centralizer of \( rM \) and let \( M' \) and \( M'' \) be \( \mathcal{C} \)-submodules of \( M_{\mathcal{C}} \) such that the image of every \( R \)-homomorphism \( rN \to rM \) is contained in \( M' \), whereas \( M'' \) is contained in the kernel of every \( R \)-homomorphism \( rM \to rN \). Then, given a \( \mathcal{C} \)-homomorphism \( \psi \) of the form

\[
M_{\mathcal{C}} \xrightarrow{\xi} M/M' \to M'' \xrightarrow{\iota} M_{\mathcal{C}}
\]

(where \( \xi \) is the canonical epimorphism, \( \iota \) the inclusion), the trivial extension

\[
\begin{bmatrix}
\psi & 0 \\
0 & 0
\end{bmatrix}: M \oplus N \to M \oplus N
\]

of \( \psi \) belongs to the double centralizer of \( r(M \oplus N) \).

If, for a module \( M \), there exists an exact sequence of \( R \)-modules

\[
0 \to M \to D_1 \to D_2 \to \ldots \to D_n
\]

with \( D_1 \) both projective and injective, then the dominant dimension \( \text{dom dim} \ M \) of the module \( M \) is \( \geq n \). Now \( \text{dom dim} \ R \geq 1 \) if and only if \( R \) is a \( QF - 3 \) ring [5]. In this case, \( \text{dom dim} \ R \geq 2 \) if and only if the minimal faithful left \( R \)-module is balanced [7]. Since the minimal faithful left \( R \)-module of a \( QF - 3 \) ring is both projective and injective, all faithful left or right modules which are either projective or injective are balanced [4, Theorem 5]. In particular, also the minimal faithful right module is balanced, and \( \text{dom dim} \ R \geq 2 \). So we simply may say that the dominant dimension of \( R \) is \( \geq 2 \).

If there exists a natural number \( m \) such that for every exact sequence of left \( R \)-modules

\[
0 \to K \to P_{m-1} \to \ldots \to P_1 \to P_0 \to M \to 0
\]

with \( P_i \) projective for \( 0 \leq i \leq m - 1 \), \( K \) is also projective, then the smallest such \( m \) is called the left global dimension of \( R \). It is easy to see that the left global dimension of \( R \) is \( \leq 2 \) if and only if the kernel of every \( R \)-homomorphism \( \overline{R}F \to \overline{R}F' \), with \( \overline{R}F \) and \( \overline{R}F' \) both free, is projective.

2. The aim of this section is to prove the following general result on \( QF - 1 \) rings.

**Proposition.** Consider a \( QF - 1 \) ring \( R \) with left socle \( L \) and right socle \( J \). Let \( e \) and \( f \) be primitive idempotents. If \( y \) is an element of \( fJe \) which does not belong to \( L \), and if \( fL \neq 0 \), then \( Ry = Je \).

**Proof.** Obviously, we may assume that \( R \) is a basis ring, because if the propo-
sition holds for a basis subring of $R$, it is also true for $R$. Also, we may assume that $y \in W$, since otherwise the conclusion is trivial.

Let $e_1$ be a primitive idempotent such that $e_1$ and $e_2 = e$ are either orthogonal or equal, and which satisfies $f(L \cap J)e_1 \neq 0$. Let $x$ be a non-zero element in $f(L \cap J)e_1$. Since $xR \cap yR = 0$, the left $R$-module

$$R^M = (Re_1 \oplus Re_2)/R(x, y)$$

is indecomposable [9]. The endomorphisms of $R^M$ are induced by matrices

$$
\begin{bmatrix}
    r_{11} & r_{12} \\
    r_{21} & r_{22}
\end{bmatrix}
$$

with entries $r_{ij} \in e_i Re_j$, for $1 \leq i, j \leq 2$, operating on $Re_1 \oplus Re_2$ from the right. If $(r_{ij})$ induces an endomorphism of $R^M$, then $r_{21}$ belongs to the radical $W$ of $R$. For, consider the image of $(x, y)$ under $(r_{ij})$. We have

$$(xr_{11} + yr_{21}, xr_{12} + yr_{22}) = (\lambda x, \lambda y)$$

for some $\lambda \in R$. Thus $yr_{21} = \lambda x - xr_{11} \in L$, and, since $y \in L$, we conclude that $r_{21} \in W$.

Also, if $(r_{ij})$ induces a nilpotent endomorphism of $R^M$, then $r_{22} \in W$. For, consider the image of $(0, y)$ under $(r_{ij})$. We have

$$
(0, y) \begin{bmatrix}
    r_{11} & r_{12} \\
    r_{21} & r_{22}
\end{bmatrix} = (yr_{21}, yr_{22}) = (0, yr_{22}),
$$

since $y \in J$ and $r_{21} \in W$. By induction, we get for natural $n$

$$
(0, y) \begin{bmatrix}
    r_{11} & r_{12} \\
    r_{21} & r_{22}
\end{bmatrix}^n = (0, yr_{22}^n).
$$

Since, by assumption, $(r_{ij})$ induces a nilpotent endomorphism, there is some $n$ with

$$(0, yr_{22}^n) = (\lambda x, \lambda y),$$

where $\lambda$ can be chosen in $Rf$. But $\lambda x = 0$ implies $\lambda \in W$, thus $\lambda$ is nilpotent. If $\lambda^n = 0$, then $yr_{22}^n = \lambda y$ yields $yr_{22}^{nm} = \lambda^m y = 0$, and consequently, $r_{22}$ cannot be invertible in $e_2 Re_2$.

Let $C$ be the centralizer of $R^M$. It follows from the considerations above that $(0 \oplus Je_2) + R(x, y)/R(x, y)$ is contained in $\text{Soc } M_C$. For, if $W'$ denotes the radical of $C$, the elements of $W'$ can be lifted to matrices $(r_{ij})$ with $r_{21}$ and $r_{22}$ in $W$. Thus, for $z \in Je_2$, we have

$$
(0, z) \begin{bmatrix}
    r_{11} & r_{12} \\
    r_{21} & r_{22}
\end{bmatrix} = (zr_{21}, zr_{22}) = (0, 0),
$$

and thus $(0, z) + R(x, y) \in \text{Soc } M_C$.

Also, $(0 \oplus Je_2) + R(x, y)/R(x, y)$ belongs to the kernel of every homo-
morphism \( \mathfrak{r} M \to R(1 - e_1) \). For, we may lift such a morphism to

\[
\begin{bmatrix}
 r_1 \\
r_2
\end{bmatrix} : Re_1 \oplus Re_2 \to R(1 - e_1)
\]

with \( r_1 \in e_1 R(1 - e_1) \), mapping \((x, y)\) into \( 0 \). The last condition gives us the equality \( x r_1 + y r_2 = 0 \), thus, since \( x \in J \) and \( r_1 \in e_1 R(1 - e_1) \subseteq W \), we get \( y r_2 = 0 \). This shows that not only \( r_1 \) but also \( r_2 \) belongs to \( W \), and, as a consequence, the image of \((0, z) \in 0 \oplus J e_2 \) under \( \begin{bmatrix} r_1 \\
r_2
\end{bmatrix} \) is \( z r_1 + z r_2 = 0 \).

Since \( x, y \in J \), every matrix

\[
\begin{bmatrix}
 r_{11} & r_{12} \\
r_{21} & r_{22}
\end{bmatrix}
\]

with \( r_{ij} \in e_1 We_j \)

induces a nilpotent endomorphism of \( \mathfrak{r} M \), thus \( We_1 \oplus We_2/R(x, y) \subseteq M W \). Moreover, if \( e_1 \) and \( e_2 \) are orthogonal, we have the equality

\[
We_1 \oplus We_2/R(x, y) = M W.
\]

For, assume that \( \begin{bmatrix} r_{11} & r_{12} \\
r_{21} & r_{22}
\end{bmatrix} \) with \( r_{ij} \in e_1 Re_j \) induces an endomorphism \( \varphi \) of \( \mathfrak{r} M \); then \( r_{12} \in e_1 Re_2 \subseteq W \), and, if \( \varphi \) is nilpotent, we conclude similarly to a proof above that

\[
(x, 0) \begin{bmatrix}
 r_{11} & r_{12} \\
r_{21} & r_{22}
\end{bmatrix}^n = (x r_{11}^n, 0),
\]

and that therefore also \( r_{11} \in W \). This shows that for \( \varphi \in W \), all \( r_{ij} \)'s belong to \( W \), so \( M W \subseteq We_1 \oplus We_2/R(x, y) \).

Next, we claim that \((e_1, 0) + R(x, y)\) does not belong to \( M W = \text{Rad} M_\mathfrak{r} \). This is obvious in the case where \( e_1 \) and \( e_2 \) are orthogonal. So, we only consider the case \( e = e_1 = e_2 \). If we assume that \((e, 0) + R(x, y)\) belongs to \( M W \), then, since \( M W \) is a proper \( R \)-submodule of \( \mathfrak{r} M \) also containing \( We \oplus We/R(x, y) \), we have \( M W = Re \oplus We/R(x, y) \). Also, \( \text{Soc} M_\mathfrak{r} \) is an essential \( \mathfrak{c} \)-submodule of \( M \), thus \((Je \oplus Je) + R(x, y)/R(x, y)\) intersects \( \text{Soc} M_\mathfrak{r} \) nontrivially. Therefore, there is a non-zero \( \mathfrak{c} \)-homomorphism \( \psi \) of the form

\[
M_\mathfrak{r} \hookrightarrow M/M W \to (Je \oplus Je) + R(x, y)/R(x, y) \overset{\iota}{\to} M_\mathfrak{r},
\]

where \( \iota \) is the canonical epimorphism, \( \iota \) the embedding. The image of every \( R \)-homomorphism \( R(1 - e) \to \mathfrak{r} M \) is contained in \( We \oplus We/R(x, y) \subseteq M W \), since we may lift such a morphism to

\[
R(1 - e) \rightarrow \begin{bmatrix} r_1 \\
r_2
\end{bmatrix} Re \oplus Re
\]

with \( r_1 \in (1 - e)Re \subseteq W \). On the other side, \((Je \oplus Je) + R(x, y)/R(x, y)\) is contained in the kernel of every morphism \( \mathfrak{r} M \to R(1 - e) \). Thus the trivial extension \( \psi' \) of \( \psi \) to \( \mathfrak{r} M \oplus R(1 - e) \) belongs to the double centralizer of
$R M \oplus R(1 - e)$. But this morphism $\psi'$ vanishes on $M \oplus R(1 - e)$ which is a faithful module since $Re$ is embeddable in $(Re \oplus We)/R(x, y) = M \oplus W$. This shows that $\psi'$ cannot be induced by multiplication, a contradiction. So we have shown that $(e, 0) + R(x, y)$ cannot belong to $M \oplus W$.

There is a $\mathcal{C}$-submodule $M'$ of $M$ which contains $M \oplus W$ and also the images of all $R$-homomorphisms $R(1 - e_1) \to R M$, but which does not contain the element $(e_1, 0) + R(x, y)$. For, in the case where $e_1$ and $e_2$ are orthogonal, choose $M' = (W e_1 \oplus Re_2)/R(x, y)$. Since all matrices $\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$ which induce endomorphisms of $R M$ satisfy $r_{12}, r_{21} \in W$, we see that $M'$ is actually a $\mathcal{C}$-submodule. Obviously, $M' \supseteq M \oplus W = W e_1 \oplus W e_2/R(x, y)$, and given an $R$-homomorphism $R(1 - e_1) \to R M$, we may lift it to

$$R(1 - e_1) \overset{(r_1, r_2)}{\to} R e_1 \oplus R e_2$$

with $r_1 \in (1 - e_1) R e_1$. But $r_1 \in (1 - e_1) R e_1 \subseteq W$, thus the image of $(r_1, r_2)$ is contained in $W e_1 \oplus R e_2$. Secondly, consider the case $e_1 = e_2$. In this case, let $M' = M \oplus W$. Since every $R$-homomorphism $R(1 - e_1) \to R M$ again can be lifted to $(r_1, r_2)$ where now both $r_1$ and $r_2$ belong to $(1 - e_1) R e_1 \subseteq W$, the image of $R(1 - e_1) \to R M$ has to be contained in

$$W e_1 \oplus W e_2/R(x, y) \subseteq M \oplus W = M'.$$

So we see that also in the second case $M'$ satisfies all conditions.

Also, there is a $\mathcal{C}$-submodule $M''$ of $M_\psi$ contained in $\text{Soc } M_\psi$ and in the kernel of every $R$-homomorphism $R M \to R(1 - e_1)$, and containing

$$(0 \oplus Je_2) + R(x, y)/R(x, y).$$

For, we simply may take the intersection of $\text{Soc } M_\psi$ and the kernels of all maps $R M \to R(1 - e_1)$.

By construction, $M'/M'$ and $M''$ both are semisimple $\mathcal{C}$-modules. Given $z \in Je_2$, there is a $\mathcal{C}$-homomorphism $\psi$ of the form

$$M_\psi \overset{\iota}{\to} M'/M' \overset{\iota}{\to} M'' \overset{\iota}{\to} M_\psi$$

(where again $\iota$ denotes the canonical epimorphism, $\iota$ the embedding) mapping $(e_1, 0) + R(x, y)$ onto the element $(0, z) + R(x, y)$. Since the image of every morphism $R(1 - e_1) \to R M$ is contained in $M'$ and the kernel of every morphism $R M \to R(1 - e_1)$ contains $M''$, the trivial extension of $\psi$ to $R M \oplus R(1 - e_1)$ belongs to the double centralizer of $R M \oplus R(1 - e_1)$. Using the fact that $R$ is a QF-1 ring, we find an element $\rho \in R$ which induces this extension. In particular, we have

$$\rho(e_1, 0) - (0, z) \in R(x, y).$$

Thus $z \in R y$, as we wanted to prove.
3. The main theorem. The result of the previous section can be considered as a forth socle condition for QF - 1 rings. Using these socle conditions we can show

**Theorem.** Let R be a QF - 1 ring and assume that the right socle J of R, considered as a left module, is projective. Then R is a QF - 3 ring.

**Proof.** Obviously, we may assume that R is two-sided-indecomposable, i.e. that there are not two two-sided non-zero ideals I_1 and I_2 with R = I_1 ⊕ I_2. Let e and f be primitive idempotents with f(L \cap J)e ≠ 0. Then according to the second socle condition

\[ \partial R Le \times \partial f J_R \leq 2. \]

We have to show that in our case the product actually is equal to 1. So, assume \( \partial R Le = 2 \) and consider first the case \( Le \subseteq Je \). The third socle condition implies \( Le = Je \). Since \( Je \) is a projective left R-module, and \( Je \) is properly contained in \( Re \), we find a non-zero idempotent \( e' \) such that \( e \) and \( e' \) are orthogonal, \( Re' \) is isomorphic to a direct summand of \( Je \), and \( Je' \neq 0 \). Then \( fL \supseteq f(L \cap J)e \oplus fLe' \) and therefore \( \partial f L_R > 1 \), a contradiction to the first socle condition. If \( Le \not\subseteq Je \), take a primitive idempotent \( f' \) and an element \( x = f'xe \in Le \setminus Je \). Let \( e' \) be a primitive idempotent and \( w = we' \in W \) with \( 0 \neq xw \in L \cap J \). Then \( \partial f' L_R > 1 \), thus, using the fact that \( f'(L \cap J)e \neq 0 \) the first socle condition implies \( \partial R Je' = 1 \). As a consequence, \( R x w = Je' \) is projective and since it is isomorphic to \( R f'/Wf' \), we conclude \( Wf' = 0 \), thus \( f' \) belongs to \( L \). But since \( x \in f'Le \setminus J \) and \( Je \neq 0 \), we may apply the Proposition of section 2 to the opposite ring of \( R \) in order to conclude that \( xR = f'L \), and therefore we find \( \rho \in R \) with \( f' = x \rho = f'x \rho \). Right multiplication by \( x \) gives an isomorphism \( R f' \rightarrow Re \). But obviously \( Re \not\subseteq L \), whereas \( R f' \subseteq L \). This contradiction proves that \( \partial R Le = 1 \).

Secondly, assume \( \partial f J_R = 2 \). If \( fJ \subseteq fL \), then according to the first socle condition we have \( \partial R Je = 1 \) for every primitive idempotent \( e \) with \( fJe = 0 \). Thus \( fJ \) is a direct summand of \( R J \), and therefore also projective. This yields that \( R f' \) is of length 1, that is \( f \in L \). But the socle condition (3*) implies \( fL \subseteq fJ \), thus \( R f' \subseteq L \cap J \). Since \( R \) is assumed to be two-sided-indecomposable, we have \( R = R f' \), and \( R \) is semisimple; but then \( \partial f R_R = 1 \), a contradiction.

Next, assume \( fJ \not\subseteq fL \), and take a primitive idempotent \( e' \) and an element \( y = fye' \in fJe \setminus L \). By the result of section 2, \( Ry = Je' \), since we assume \( f(L \cap J)e \neq 0 \). Now, if \( Je' \) is a proper submodule of \( Re' \), then using the fact that \( Re' \) is projective and local, we find a primitive idempotent \( e'' \), orthogonal to \( e' \), with \( Je' = Re' \). If \( f' \) is a primitive idempotent with \( f'(L \cap J)e' \neq 0 \), then also \( f'Le' \neq 0 \), thus \( \partial f' L_R > 1 \). But since \( Je \not\subseteq L \), we also have \( \partial R Je' > 1 \). Together with \( f'(L \cap J)e' \) this gives a contradiction to the first socle condition. So, we have to assume that \( Je' = Re' \). Since \( Ry = Je' \) and \( y = fye' \), we may assume \( e' = f \). Now \( R f \subseteq J \), and \( f \not\in L \), thus no simple left ideal can be isomorphic to \( R f/Wf' \). But this is a contradiction to \( fL \neq 0 \), and therefore we have shown \( \partial f J_R = 1 \).
COROLLARY. A QF — 1 ring of left global dimension \( \leq 2 \) is a QF — 3 ring.

Proof. Let \( R \) be a QF — 1 ring of left global dimension \( \leq 2 \). If \( w_1, \ldots, w_n \) are generators of \( W_R \), consider the maps

\[
\varphi : R^R \rightarrow \bigoplus_{i=1}^n R^R
\]

with \( 1 \varphi = (w_1, \ldots, w_n) \). Then the right socle \( J \) of \( R \) is just the kernel of \( \varphi \), so \( R^J \) has to be projective.

4. Remarks. If we consider the class of rings of left global dimension \( \leq 2 \), we asked in the introduction for a characterization of those rings \( R \) with \( \text{dom dim} \ R \geq 2 \). The following example shows that not all rings of global dimension \( \leq 2 \) and dominant dimension \( \geq 2 \) are QF — 1 rings.

Let \( R \) be a generalized uniserial ring with the Kupisch series

\[
1, 2, 2, 3, 2.
\]

Then, according to [3], \( R \) is not a QF — 1 ring, but since \( R \) is generalized uniserial and coincides with its complete ring of left quotients, \( \text{dom dim} \ R \geq 2 \). Also, the global dimension of \( R \) is 2.

On the other side, the QF — 1 rings of global dimension \( \leq 2 \) are not all of dominant dimension \( \geq 3 \), as Morita’s second example in [6] shows. It can easily be seen that the dominant dimension of this algebra is precisely 2.

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