THE INTERSECTION PROPERTY OF AMALGAMATIONS

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Let $M$ denote a class of monomorphisms in a category, with the following properties:

(i) If $fg \in M$ and $f \in M$ then $g \in M$.
(ii) The intersection of two $M$'s is an $M$.
(iii) The pushout of an $M$ by an $M$ exists and is an $M$.

The first two properties are enjoyed by the class of all monomorphisms, by the class of regular monomorphisms, and again by the distinguished class of monomorphisms in an Isbell bicategory structure. The third is the so-called amalgamation property. There has been some interest of late in categories with the amalgamation property, and in particular in such categories wherein the pushout of two $M$'s is also a pullback (= intersection). We give a necessary and sufficient condition for the latter to be the case.

Proposition. In the diagram below, let $m_1 \in M$, let $n_1$, $n_2$ be the pushout of $m_1$, $m_2$, let $p_1$, $p_2$ be the pullback of $n_1$, $n_2$, and let $z$ be the unique map rendering the diagram commutative. Then $z$ is an epimorphism.

\[ \begin{array}{ccc}
  m_1 & \xrightarrow{p_1} & n_1 \\
  \downarrow{z} & & \downarrow{z} \\
  m_2 & \xrightarrow{p_2} & n_2
\end{array} \]

Proof. The $n_i \in M$ by (iii), the $p_i \in M$ by (ii) and (i), and $z \in M$ by (i). Form the
in which every diamond is a pushout. Then the exterior of (2) coincides with that of (1), so we may assume that $q_1w_i = n_i$. Because $z$ and the $p_i$ are $M$'s, every map in (2) is an $M$ by (iii) (and the pushouts do exist). Since

$$q_1v_1u_1 = q_1w_1p_1 = n_1p_1 = n_2p_2 = q_2v_2u_2 = q_1v_1u_2,$$

and since $q_1$ and $v_1$ are both monomorphisms, we have $u_1 = u_2$. Since $u_1, u_2$ is the cokernel-pair of $z$, it follows that $z$ is an epimorphism.

**Corollary.** Let the category admit intersections of $M$'s. Then every pushout of two $M$'s is also a pullback if and only if every $M$ that is an epimorphism is in fact an isomorphism.

**Proof.** The “if” part follows from the proposition. For the “only if” part let $m$ be an epimorphism in $M$. Taking $m_1 = m_2 = m$ in diagram (1), we have $n_1 = n_2 = 1$, whence $p_1 = p_2 = 1$, whence $z = m$; and, by assumption, $z$ is an isomorphism.

**Remark.** The condition in the corollary is automatic if $M$ is the extremal monomorphisms or the regular monomorphisms. If $M$ is the monomorphism, it says that every bimorphism (= epimorphic monomorphism) is an isomorphism. Suppose (i) holds in the stronger form:

If $fg \in M$, then $g \in M$. Then, for a fixed $m_1$, the $z$ in (1) will always be an isomorphism if and only if $m_1$ is an extremal monomorphism.

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