THE HOCHSCHILD COCYCLE CORRESPONDING TO A LONG EXACT SEQUENCE

Dedicated to Hiroyuki Tachikawa on his 60th birthday

By

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1. Let \( k \) be a field, and \( A \) an associative \( k \)-algebra with 1. Let \( M, N \) be right \( A \)-modules. We denote by \( H^i \) the Hochschild cohomology of \( A \). It is well-known that there is a natural isomorphism

\[
\eta_{M,N} : \Ext^i_k(M, N) \to H^i(A, \Hom_k(M, N))
\]

see Cartan-Eilenberg [CE], Corollary IX. 4.4. For \( t \geq 1 \), the elements of \( \Ext^i_k(M, N) \) may be considered as equivalence classes of long exact sequences, see Mac Lane [M], chapter III. Let

\[
E = (0 \to M \to Y_1 \to Y_2 \to \cdots \to Y_t \to N \to 0)
\]

be an exact sequence. We want to derive a recipe for obtaining a corresponding cocycle \( A^{(t+2)} \to \Hom_k(M, N) \).

For \( 0 \leq i \leq t+1 \), let \( Z_i \) be right \( A \)-modules, and for \( 0 \leq i \leq t \), let \( \beta_i : Z_i \to Z_{i+1} \) be \( k \)-linear maps. With \( \beta = (\beta_0, \ldots, \beta_t) \) we associate a map

\[
\Omega_{\beta} : A^{(t+2)} \to \Hom_k(Z_0, Z_{t+1})
\]

defined by

\[
(a_0, \ldots, a_{t+1}) \Omega_{\beta} = \tilde{a}_0 \beta_0 \tilde{a}_1 \beta_1 \cdots \tilde{a}_t \beta_t \tilde{a}_{t+1},
\]

for \( a_0, \ldots, a_{t+1} \in A \), where \( \tilde{a}_i \) denotes the scalar multiplication by \( a_i \) (on \( Z_i \)); note that all maps will be written on the right of the argument, thus the composition of \( \beta_0 : Z_0 \to Z_1 \), and \( \beta_1 : Z_1 \to Z_2 \) is denoted by \( \beta_0 \beta_1 \).

Given the exact sequence \( E \) exhibited above, it clearly splits as a sequence of \( k \)-spaces, thus there are \( k \)-linear maps

\[
M \to Y_1 \to Y_2 \to \cdots \to Y_t \to N
\]

such that

\[
\gamma_{i-1} \gamma_i = 0, \quad \gamma_{i-1} \gamma_i + \gamma_i g_i = 1_{Y_i}, \quad \text{for } 1 \leq i \leq t,
\]

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and 
\[ \gamma_0 g_0 = 1_V, \quad g_i \gamma_i = 1_V, \]
(see section 2).

**Theorem.** The map \( \Omega_i : A^{op} \otimes A \to \text{Hom}_A(M, N) \) is a cocycle, and the cohomology classes \([\Omega_i]\) and \(\eta([E])\) in \(H^i(A, \text{Hom}_A(M, N))\) are equal up to sign.

One reason for our interest in this problem is the following: Consider the case \( i = 2 \). Given any bimodule \( A \otimes A \), the elements of \( H^i(A, T) \) index the various "Hochschild extensions" \( \tilde{A} \) of \( A \) by \( T \) (here, \( \tilde{A} \) is a \( k \)-algebra with a square zero ideal \( I \) such that \( \tilde{A}/I = A \), and such that \( I \), as an \( A-A \)-bimodule, is isomorphic to \( T \); note that the multiplication of \( \tilde{A} \) can be recovered from \( A \) and \( T \) using the corresponding 2-cocycle, see [H] or [CE], XIV. 2). There is a recursive construction for quasi-hereditary algebras due to Parshall and Scott ([PS], Theorem 4.6) which uses Hochschild extensions of quasi-hereditary algebras \( A \) by bimodules of the form \( \text{Hom}_A(M, N) \), so we have to deal with 2-cocycles \( A^{op} \otimes A \to \text{Hom}_A(M, N) \). Our presentation of such 2-cocycles using long exact sequences should help to understand these algebras. Also, we remark that the Hochschild cohomology groups with values in \( \text{Hom}_A(DA, A) \), where \( DA = \text{Hom}_A(A, k) \), play a prominent role in Tachikawa's discussion of the Nakayama conjecture [T].

2. The splitting for \( E \) over \( k \). In order to work with the sequence \( E \), it will be convenient to use the notation: \( Y_{-1} = 0, \ Y_0 = M, \ Y_{t+1} = N, \ Y_{t+2} = 0 \), and to deal also with the zero maps \( g_{-1} : Y_0 \to Y_{-1}, \ \gamma_{-1} : Y_{-1} \to Y_0, \ \gamma_{t+1} : Y_{t+2} \to Y_{t+1}, \ \gamma_{t+2} : Y_{t+1} \to Y_{t+2} \); so that the conditions mentioned above can be rewritten in the form
\[ \gamma_{t-1} \gamma_t = 0, \quad g_i \gamma_{t-1} + \gamma_t g_i = 1_{Y_i}, \quad \text{for } 0 \leq i \leq t+1. \]

Let \( X_i \) be the image of \( g_i \), thus we have short exact sequences
\[ \begin{array}{cccccccc}
0 & \to & X_{i-1} & \xrightarrow{h_{i-1}} & Y_i & \xrightarrow{f_i} & X_i & \to & 0
\end{array} \]
for \( 1 \leq i \leq t \), with \( g_0 = h_0, \ g_i = h_i f_i \) for \( 1 \leq i \leq t-1 \), and \( g_t = f_t \). These sequences split over \( k \), thus we obtain \( k \)-linear maps \( \varphi_i : Y_i \to X_i, \ \eta_{i-1} : X_{i-1} \to Y_i \) such that
\[ \begin{array}{c}
\gamma_i \varphi_i = 0, \quad f_i \varphi_i = 1_{X_i}, \quad \eta_i h_{i-1} = 1_{X_{i-1}}, \quad \text{and } h_{i-1} \eta_{i-1} + \varphi_i f_i = 1_{Y_i}, \quad \text{for all } i.
\end{array} \]
Now, take \( \gamma_i = \varphi_i \eta_i : Y_i \to Y_{i+1} \), in this way we obtain a splitting of \( E \) over \( k \).

3. Preparation for the proof. Let \( A^e = A^{op} \otimes_k A \) be the enveloping algebra.
of $A$, where $A^{op}$ is the opposite algebra of $A$. The $A-A$-bimodules are just the (right) $A^e$-modules, in particular, $A$ itself is in a canonical way an $A^e$-module. For $n \geq 0$, let $S_n = A^{op}(n+1)$, and let $\nabla_n : S_{n+1} \rightarrow S_n$ be defined by

$$(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) \nabla_n = \sum_{i=0}^{n+1} (-1)^i a_0 \otimes \cdots \otimes (a_i a_{i+1}) \otimes \cdots \otimes a_{n+2}.$$ 

Also, let $\nabla_{-1} : S_0 \rightarrow A$ be defined by

$$(a_0 \otimes a_1) \nabla_{-1} = a_0 a_1.$$ 

The $S_n$ are $A-A$-bimodules, or, equivalently $A^e$-modules, the scalar multiplication of $a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1} \in S_n$ by $a \otimes a' \in A^{op} \otimes A = A^e$ yields $(aa_0) \otimes a_1 \otimes \cdots \otimes (a_{n+1} a')$.

Note that for all $n \geq -1$, the maps $\nabla_i$ are $A^e$-linear, in fact

$$\nabla_{-1} - S_0 \rightarrow \cdots \nabla_0 \rightarrow A$$

is a projective resolution of $A$ as a right $A^e$-module, it is called the standard resolution of $A$, see [CE], IX. 6. We can use this resolution in order to calculate $H^i(A, \text{Hom}_A(M, N)) = \text{Ext}_A^i(A, \text{Hom}_A(M, N))$.

4. Besides $\gamma = (\gamma_0, \cdots, \gamma_r)$, we also will need for $0 \leq r \leq t$, the sequences $\gamma(r) = (\gamma_0, \cdots, \gamma_r)$, so that $\gamma(0) = (\gamma_0)$, $\gamma(t) = \gamma$. According to section 1, there is defined $\Omega_{\gamma(r)} : S_r \rightarrow \text{Hom}_A(Y_0, Y_{r+1})$. In addition, by abuse of language, we also define $\Omega_{\gamma(r-1)} : A \rightarrow \text{Hom}_A(Y_0, Y_0)$ by $a \Omega_{\gamma(r-1)} = a$, for $a \in A$.

**LEMMA.** For $0 \leq r \leq t$, we have $\nabla_{r-1} \Omega_{\gamma(r-1)} = (-1)^r \Omega_{\gamma(r)} \text{Hom}(1, g_r)$.

**PROOF.** We introduce the following notation: let $\sigma_i = \gamma_i d_{i+1} \cdots d_t$, $\tau_i = \gamma_i \gamma_{i+1} \cdots \gamma_t$ for $0 \leq i \leq t-1$, and let $\sigma_{ij} = \sigma_{i-1} \cdots \sigma_j$, $\tau_{ij} = \tau_{i-1} \cdots \tau_j$ for $0 \leq i \leq j \leq t-1$; by abuse of language, let $\sigma_{i+1, i} = 1_{Y_0}$ and $\tau_{t+1, i} = 1_{Y_{t+1}}$. Recall that

$$(a_0 \otimes \cdots \otimes a_r) \nabla_{r-1} = \sum_{i=0}^r (-1)^i a_0 \otimes \cdots \otimes (a_{i-1} a_i) \otimes \cdots \otimes a_r ,$$

thus

$$(a_0 \otimes \cdots \otimes a_r) \nabla_{r-1} \Omega_{\gamma(r-1)} = \sum_{i=0}^r (-1)^i d_{i-1} \sigma_{0, i-1} \tau_{i, r-1} d_r$$

$$= \sum_{i=0}^r (-1)^i d_{i-1} \sigma_{0, i-1} g_{i-1} \gamma_{i-1} + \gamma_i g_i \tau_{i, r-1} d_r ,$$

where we have inserted $1_{Y_0} = g_{i-1} \gamma_{i-1} + \gamma_i g_i$. Note that for $0 \leq i \leq r-1$, we have

$$\sigma_{0, i-1} \gamma_i g_i \tau_{i, r-1} = \sigma_{0, r-1} \gamma_i g_i d_r \gamma_i \tau_{i+1, r-1}$$

$$= \sigma_{0, r-1} \gamma_i g_i \tau_{i+1, r-1}$$

$$= \sigma_{0, i} g_i \tau_{i+1, r-1} .$$
since $g_i$ is $A$-linear. As a consequence, the last term of the summand with index $i$ and the first term of the summand with index $i+1$ are equal up to sign, so they cancel. In addition, the first term of the summand with index $i=0$ involves $g_{-1}=0$, thus vanishes. It remains

\[
(a_0 \otimes \cdots \otimes a_r)\nabla_{r-1} Q_{(r-1)} = (-1)^r a_{-1} g_{r-1} g_r a_r = (-1)^r a_{-1} g_{r-1} g_r \]

\[
= (-1)^r (a_0 \otimes \cdots \otimes a_r) Q_{(r)} - g_r = (-1)^r (a_0 \otimes \cdots \otimes a_r) Q_{(r)} \text{Hom}(1, g_r).
\]

This finishes the proof.

5. An injective coresolution of the $A$--$A$-bimodule $\text{Hom}_A(M, N)$. We choose a projective resolution

\[
0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots
\]

of the $A$-module $M$, and an injective coresolution

\[
0 \rightarrow N \rightarrow Q^0 \rightarrow Q^1 \rightarrow \cdots
\]

of the $A$-module $N$. For $i \geq 0$, let $L^i = \bigoplus_{i=0}^t \text{Hom}_A(P_i, Q^{i-i})$, this is an $A$--$A$-bimodule, or, equivalently a right $A'$-module. For $i \geq 0$, define an $A'$-linear map $\Delta^i : L^i \rightarrow L^{i+1}$ by

\[(\varphi_0, \ldots, \varphi_i) \Delta^i = (\varphi_i q^i, (-1)^{i+1} p_i \varphi_{i-1} + \varphi_i q^{i-1}, \ldots, (-1)^i p_{i-1} \varphi_{i-1} + \varphi_i q^i, (-1)^{i+1} p_i \varphi_{i}),\]

where $\varphi_i \in \text{Hom}_A(P_i, Q^{i-i})$, and define $\Delta^{-1} : \text{Hom}_A(M, N) \rightarrow L^0$ by $\Delta^{-1} = \text{Hom}(p_{-1}, q^{-1})$. We obtain a sequence

\[
0 \rightarrow \text{Hom}_A(M, N) \rightarrow L^0 \rightarrow L^1 \rightarrow \cdots,
\]

which is an injective coresolution, see [CE], IX, Cor. 2.7a.

In order to relate the given sequence $E$ with the injective coresolution $Q^i = (Q^i, q^i)$, we define $u_{-i} = 1_N$, and, inductively, we find $u_i : Y_{i-i} \rightarrow Q^i$ such that $g_{i-i} u_{i-i} = q_i q_i^{-1}$, for $0 \leq i \leq t$.

We are going to reformulate the previous lemma using the maps $\Delta^i$ and $u_i$. For $0 \leq r \leq t-1$, let

\[
Q_r^i : S_r \rightarrow L^{i-r-1}
\]

be defined by

\[
(a_0 \otimes \cdots \otimes a_{r+1}) Q_r^i = (p_{-r}(a_0 \otimes \cdots \otimes a_{r+1}(Q_{(r)}, u_{i-r-1}, 0, \ldots, 0)),
\]
and similarly, let

$$Q'_{-1}: A \longrightarrow L'$$

be defined by

$$(a)Q'_{-1} = (p_{-1}a, 0, \cdots, 0).$$

**Proposition.** For $0 \leq r \leq t-1$, we have $\nabla_{r-1}Q'_{r-1} = (-1)^rQ_r\Delta^{t-r-1}$. For $r=t$, we have $\nabla_{t-1}Q'_{t-1} = (-1)^{r}Q_r\Delta^{-1}$.

**Proof.** For $0 \leq r \leq t$, and $a_0, \cdots, a_{t+1} \in A$, we have

$$(a_0 \otimes \cdots \otimes a_{t+1})\nabla_{r-1}Q'_{r-1} = (p_{-1}(a_0 \otimes \cdots \otimes a_{t+1})\nabla_{r-1}Q_{r-1})u_{t-r}, 0, \cdots, 0)
\tag{1.2}$$

$$= (-1)^r(p_{-1}(a_0 \otimes \cdots \otimes a_{t+1})Q_r\Delta^{t-r-1})u_{t-r}, 0, \cdots, 0)
\tag{1.3}$$

$$= (-1)^r(p_{-1}(a_0 \otimes \cdots \otimes a_{t+1})Q_r\Delta^{t-r-1})u_{t-r-1}q^{t-r-1}, 0, \cdots, 0)
\tag{1.4}$$

using the definition of $Q'_{r-1}$, the lemma, and the defining condition for $u_{t-r}$. On the other hand, for $0 \leq r \leq t-1$, we have

$$(a_0 \otimes \cdots \otimes a_{t+1})Q_r\Delta^{t-r-1} = (p_{-1}(a_0 \otimes \cdots \otimes a_{t+1})Q_ru_{t-r-1}, 0, \cdots, 0)\Delta^{t-r-1}
\tag{1.5}$$

$$= (p_{-1}(a_0 \otimes \cdots \otimes a_{t+1})Q_ru_{t-r-1}q^{t-r-1}, 0, \cdots, 0)
\tag{1.6}$$

using the definitions of $Q_r, \Delta^{t-r-1}$, and the fact that $p_{-1}p_{-1}=0$. Similarly, for $r=t$, we have

$$(a_0 \otimes \cdots \otimes a_{t+1})Q_t\Delta^{t-1} = p_{-1}(a_0 \otimes \cdots \otimes a_{t+1})Q_tq^{-1}
\tag{1.7}$$

$$= p_{-1}(a_0 \otimes \cdots \otimes a_{t+1})Q_{t-1}u_{t-1}q^{-1}
\tag{1.8}$$

since $Q_t=Q_{t-1}$ and $u_{t-1}=1$.

6. *Some homological algebra.* We will need some basic result of homological algebra which we want to review. We have chosen already a projective resolution of $M$, and an injective coresolution of $N$. In order to calculate $\text{Ext}^t(M, N)$ we may use one of these sequences, or else the double complex $\text{Hom}_A(P, Q^t)$. So let $R^t = \bigoplus_{i=0}^t \text{Hom}_A(P_i, Q^{t-i})$, this is a subset of $L^t = \bigoplus_{i=0}^t \text{Hom}_A(P_i, Q^{t-i})$, and let $\partial^t: R^t \rightarrow R^{t+1}$ be the restriction of $\Delta^t$ to $R^t$, similarly, let $\delta^{-1}: \text{Hom}_A(M, N) \rightarrow L^s$ be the restriction of $\Delta^{-1} = \text{Hom}(p_{-1}, q^{-1})$ to $\text{Hom}_A(M, N)$. So we obtain a complex

$$R^t = (R^0 \xrightarrow{\partial^0} R^1 \xrightarrow{\partial^1} R^2 \rightarrow \cdots),$$

which we want to compare with the complexes

$\text{Hom}_A(P, N)$ and $\text{Hom}_A(M, Q^t)$. 
Note that there are maps
\[ \text{Hom}(1, q^{-1}) : \text{Hom}_A(P, N) \longrightarrow R', \]
\[ \text{Hom}(p^{-1}, 1) : \text{Hom}_A(M, Q') \longrightarrow R', \]
and they are quasi-isomorphisms: they induce isomorphisms when passing to the cohomology ([B], §5.2).

Consider now the given exact sequence \( E \). Its equivalence class \([E]\) in \( \text{Ext}_A^i(M, N) = H^i(\text{Hom}_A(P, N)) \) is given by the cocycle \( u_t : M \Rightarrow Q_t \). Under the map \( \text{Hom}(p^{-1}, 1) : \text{Hom}_A(M, Q') \rightarrow R' \), the cocycle \( u_t \) is mapped onto the cocycle \( (p^{-1}u_t, 0, \cdots, 0) \in \bigoplus_{i=0}^t \text{Hom}_A(P_i, Q'^t) = R' \).

7. Proof of the theorem. We apply the previous considerations to the ring \( A^t \) (instead of \( A \)), and the \( A^t \)-modules \( A \) and \( \text{Hom}_A(M, N) \). For \( A \), we use the standard resolution \( S. := (S_i, \nabla_i) \), for \( \text{Hom}_A(M, N) \) we use the injective coreolution \( L' := (L', \Delta') \). We form \( C^i = \bigoplus_{i=0}^t \text{Hom}_A(S_i, L'^t) \), with differential \( D^i : C^i \rightarrow C^{i+1} \) given by
\[
(\Phi_0, \cdots, \Phi_t)D^i = (\Phi_i \Delta^i, (-1)^{i+1} \nabla_i \Phi_0 + \Phi_i \Delta^{i-1}, \cdots,
(-1)^{i+1} \nabla_{i-1} \Phi_{i-1} + \Phi_i \Delta^i, (-1)^{i+1} \nabla_i \Phi_t),
\]
for \( \Phi_i \in \text{Hom}_A(S_i, L'^t) \). The maps
\[ \text{Hom}(1, \Delta^{-1}) : \text{Hom}_A(S., \text{Hom}_A(M, N)) \longrightarrow C^i \]
and
\[ \text{Hom}(\nabla^{-1}, 1) : \text{Hom}_A(A, L') \longrightarrow C^i \]
are quasi-isomorphisms. Clearly, we have an isomorphism
\[ \rho : \text{Hom}_A(A, L') \longrightarrow R', \]
since for \( A \)-modules \( X, Y \), the bimodule maps \( \Sigma : A \rightarrow \text{Hom}_A(X, Y) \) correspond bijectively to the elements of \( \text{Hom}_A(X, Y) \), with \( (\Sigma) \rho = (1) \Sigma \).

It remains to chase elements via the various quasi-isomorphisms
\[ \text{Hom}(1, \Delta^{-1}) \text{Hom}(\nabla^{-1}, L) \]
\[ \text{Hom}_A(S., \text{Hom}_A(M, N)) \longrightarrow C^i \leftarrow \text{Hom}_A(A, L'), \]
and
\[ \text{Hom}(p^{-1}, 1) \text{Hom}_A(M, Q') \longrightarrow R' \rightarrow \text{Hom}_A(A, L'). \]

The last map \( \text{Hom}(p^{-1}, 1) \) sends the cocycle \( u_t \) onto the element \( (p^{-1}u_t, 0, \cdots, 0) \in R' \), thus to \( Q'_t \) in \( \text{Hom}_A(A, L') \). So it remains to consider the elements
in $C'$. Let $\varepsilon_{ij} = (-1)^j$, and $\varepsilon_{i+1,j} = (-1)^{i+1}$, thus $\varepsilon_j = (-1)^{j+1} \varepsilon_{j-1}$, for all $j$. Let $\Phi_i = \phi_i \Delta_i^{-1}$ for $0 \leq i \leq t-1$, and $(\Psi_0, \ldots, \Psi_t) = (\Phi_0, \ldots, \Phi_{t-1}) D^{-1}$. Then

$$\Psi_i = \phi_i \Delta_i^{-1} = \varepsilon_i \phi_i \Delta_i^{-1} = \nabla_i \phi_i \Delta_i^{-1},$$

whereas, for $1 \leq i \leq t-1$,

$$\Psi_i = (-1)^i \nabla_i \phi_i \Delta_i^{-1} + \phi_i \Delta_i^{-1} = (-1)^i \varepsilon_i \nabla_i \phi_i \Delta_i^{-1} + (-1)^i \phi_i \Delta_i^{-1} = 0,$$

always using the proposition. This shows that

$$(\nabla_i \phi_i \Delta_i^{-1}, 0, \ldots, 0, (-1)^i \varepsilon_i \phi_i \Delta_i^{-1}) = (\Phi_0, \ldots, \Phi_{t-1}) D^{-1}$$

is a coboundary in $C'$, so $\nabla_i \phi_i \Delta_i^{-1}$ and $(-1)^i \varepsilon_i \phi_i \Delta_i^{-1}$ yield the same cohomology class in $H^i(C')$.

Let us summarize: the composition of $H^i(\text{Hom}(\phi_{-1}, 1))$, $H^i(\phi_{-1})$, $H^i(\text{Hom}(\nabla_{-1}, 1))$ and $H^i(\text{Hom}(1, \Delta^{-1}))^{-1}$ yields a natural isomorphism

$$\eta_{MN}: \text{Ext}_M(N, M) \rightarrow H^i(A, \text{Hom}_A(M, N))$$

and $\eta_{MN}(E) = (-1)^{i+1} \phi_i \Delta_i^{-1}$, thus $\eta_{MN}(E)$ and $[\phi_i \Delta_i^{-1}]$ are equal up to sign. This completes the proof.

\textbf{Remark.} As the proof shows, the precise relation (under the given identification of $H^i(A, \text{Hom}_A(M, N))$ and $\text{Ext}_M(N, M)$) is

$$\eta_{MN}(E) = (-1)^{i+1} [\phi_i \Delta_i^{-1}],$$

where $i$ is the largest integer with $2i \leq t$ (for $t = 2i$, we have the sign $(-1)^{i+1} \varepsilon_{i+1} = (-1)^{i+1} = (-1)^i$, for $t = 2i + 1$, we have $(-1)^{i+1} \varepsilon_{2i+1} = (-1)^{i+1}(-1)^{i+1} = (-1)^{i+1}$).

\textbf{References}


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