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A Remark on Simultaneous Inclusions of the Zeros of a Polynomial by Gershgorin's Theorem

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Summary. By using Gershgorin's theorem and the theorems on minimal Gershgorin disks a posteriori error bounds for the zeros of a polynomial are deduced, from which the bounds given in [1] by Braess and Hadeler are easily obtained.

In a recent paper [1] Braess and Hadeler gave a posteriori error bounds for zeros of a polynomial. In their proofs ideas are used which are quite similar to these used in proving Gershgorin's theorem and the theorems on minimal Gershgorin disks [2, 3, 4, 6]. In this note it is shown that their results can be readily obtained by using the abovementioned theorems explicitly.

Let p be a polynomial of degree n with leading coefficient unity, let $x_1, \ldots, x_n$ be distinct complex numbers supposed to be approximations to the zeros of p and

$$Q(x) = (x - x_1) \ldots (x - x_n).$$

By the Lagrange interpolation formula we get

$$p(x) = Q(x)\left[1 + \sum_{j=1}^{n} \frac{p(x_j)}{(x - x_j)\, Q'(x_j)}\right].$$

Let z be a zero of p, $z \neq x_i$, $i = 1\,(1)\,n$, then

$$\sum_{j=1}^{n} \frac{p(x_j)}{Q'(x_j)} \cdot \frac{1}{z - x_j} = -1.$$

Defining $\sigma_j = p(x_j)/Q'(x_j)$ this is equivalent with

$$\frac{z}{x_j - z} = \frac{x_j - \sigma_j}{x_j - z} - \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{\sigma_j}{x_j - z} \qquad j = 1\,(1)\,n$$

or

$$A u = z u \tag{1}$$

where

$$u^T = \left(\frac{1}{x_1 - z}, \ldots, \frac{1}{x_n - z}\right), \qquad A = \operatorname{diag}(x_i) - e\sigma^T$$

with

$$\sigma^T = (\sigma_1, \ldots, \sigma_n), \qquad e^T = (1, \ldots, 1).$$

Evidently a zero $z = x_j$ is an eigenvalue of A, too.

We remark that A is diagonally similar to A^T and to the matrix $J - p h^T$ considered in [5].

Gershgorin's theorem applied to (1) yields immediately

Theorem 1. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a positive vector and

$$\Gamma_j(\alpha) \equiv \left\{x : |x - x_j + \sigma_j| \leqq \frac{1}{\alpha_j} \sum_{j \neq i} \alpha_i |\sigma_i| \right\}.$$

Then $\Gamma(\alpha) = \bigcup_{i=1}^{n} \Gamma_j(\alpha)$ contains all zeros of p. A connected component of $\Gamma(\alpha)$ consisting of m disks contains exactly m zeros of p.

Remark. $\Gamma_j(\alpha)$ is a subset of $G_j = \left\{x : |x - x_j| \leqq \frac{1}{\alpha_j} \sum_{i=1}^{n} \alpha_j |\sigma_j| \right\}$ considered in [1]. Hence Theorem 1 and the subsequent result in [1] follow.

The theory of minimal separated Gershgorin disks can be applied as well.

Let $c \in C$, I a nonvoid proper subset of $\{1, \ldots, n\}$ with s elements and z_i, $i \in I$, near to c. Let

$$\bar{d} = \min\{|x_i - \sigma_i - c|, i \notin I\}$$
$$\underline{d} = \max\{|x_i - \sigma_i - c|, i \in I\}.$$

We define $K_{c,\lambda} = \{x : |x - c| \leqq \lambda\}$.

From [3], Satz 5 we get

Theorem 2. Let $\lambda \geqq 0$. There exists $\alpha > 0$ such that

$$\Gamma_j(\alpha) \subset K_{c,\lambda} \qquad j \in I$$
$$\Gamma_j(\alpha) \cap \mathring{K}_{c,\lambda} = \emptyset \qquad j \notin I$$

iff $\lambda \in (\underline{d}, \bar{d})$ and

$$f(\lambda) = \sum_{i \in I} \frac{|\sigma_i|}{|\sigma_i| - |x_i - \sigma_i - c| + \lambda} + \sum_{i \notin I} \frac{|\sigma_i|}{|\sigma_i| + |x_i - \sigma_i - c| - \lambda} \leqq 1. \tag{2}$$

Satz 6 in [3] gives

Theorem 3. Let $|\sigma_i| \leqq \varepsilon \quad i = 1\,(1)\,n$,

$$w = \bar{d} - \underline{d} - (n - 2s)\,\varepsilon$$

and

$$\varepsilon \leqq \frac{\bar{d} - \underline{d}}{n - 2 + 2\sqrt{s(n-s)}}. \tag{3}$$

Then there are at least s roots of p in $K_{c,\lambda}$, where

$$\lambda = \underline{d} + \tfrac{1}{2}\left[w - \sqrt{w^2 - 4(n-1)\,\varepsilon^2 - 4(\bar{d} - \underline{d})\,(s-1)\,\varepsilon}\right]. \tag{4}$$

From this result Theorem 3 in [1] follows easily:

Let $|\sigma_i| \leqq \varepsilon \; i = 1\,(1)\,n$, and let c satisfy

$$|c - x_i| \leqq q, \; i \in I, \quad \sigma|c - x_i| \geqq d, \; i \notin I$$

and

$$\varepsilon \leqq \frac{d - q}{n + 2\sqrt{s(n-s)}}. \tag{5}$$

Obviously with this c

$$\underline{d} \leqq q + \varepsilon, \quad \bar{d} \geqq d - \varepsilon$$

and (3) follows from (5). By some calculations we see that the right hand side in (4) is not greater than

$$q + \tfrac{1}{2}\left[\tilde{w} - \sqrt{\tilde{w}^2 - 4s\,\varepsilon(d-q)}\right]$$

with $\tilde{w} = d - q - (n-2s)\,\varepsilon$, which is just the bound given in [1].

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